Nonstandard Concepts for Handling Imprecise Data and Imprecise Probabilities
Problems with Probability Theory

Representation of Ignorance

We are given a die with faces 1, . . . , 6
What is the certainty of showing up face \( i \) ?

- Conduct a statistical survey (roll the die 10000 times) and estimate the relative frequency: \( P(\{i\}) = \frac{1}{6} \)
- Use subjective probabilities (which is often the normal case): We do not know anything (especially and explicitly we do not have any reason to assign unequal probabilities), so the most plausible distribution is a uniform one.

Problem: Uniform distribution because of ignorance or extensive statistical tests

Experts analyze aircraft shapes: 3 aircraft types \( A, B, C \)
“It is type \( A \) or \( B \) with 90\% certainty. About \( C \), I don’t have any clue and I do not want to commit myself. No preferences for \( A \) or \( B \).”

Problem: Ignorance hard to handle with Bayesian theory
“$A \subseteq X$ being an imprecise date” means: the true value $x_0$ lies in $A$ but there are no preferences on $A$.

$\Omega$ set of possible elementary events
$\Theta = \{\xi\}$ set of observers
$\lambda(\xi)$ importance of observer $\xi$

Some elementary event from $\Omega$ occurs and every observer $\xi \in \Theta$ shall announce which elementary events she personally considers possible. This set is denoted by $\Gamma(\xi) \subseteq \Omega$. $\Gamma(\xi)$ is then an imprecise date.

$\lambda : 2^\Theta \rightarrow [0, 1]$ probability measure
(interpreted as importance measure)
$(\Theta, 2^\Theta, \lambda)$ probability space
$\Gamma : \Theta \rightarrow 2^\Omega$ set-valued mapping
Let $A \subseteq \Omega$:

a) $\Gamma^*(A) \overset{\text{Def}}{=} \{\xi \in \Theta \mid \Gamma(\xi) \cap A \neq \emptyset\}$

b) $\Gamma_*(A) \overset{\text{Def}}{=} \{\xi \in \Theta \mid \Gamma(\xi) \neq \emptyset \text{ and } \Gamma(\xi) \subseteq A\}$

Remarks:

a) If $\xi \in \Gamma^*(A)$, then it is *plausible* for $\xi$ that the occurred elementary event lies in $A$.

b) If $\xi \in \Gamma_*(A)$, then it is *certain* for $\xi$ that the event lies in $A$.

c) $\{\xi \mid \Gamma(\xi) \neq \emptyset\} = \Gamma^*(\Omega) = \Gamma_*(\Omega)$

Let $\lambda(\Gamma^*(\Omega)) > 0$. Then we call

$$P^*(A) = \frac{\lambda(\Gamma^*(A))}{\lambda(\Gamma^*(\Omega))}$$

the upper, and

$$P_*(A) = \frac{\lambda(\Gamma_*(A))}{\lambda(\Gamma_*(\Omega))}$$

the lower probability w. r. t. $\lambda$ and $\Gamma$. 
Example

$$\Theta = \{a, b, c, d\}$$
$$\Omega = \{1, 2, 3\}$$
$$\Gamma^*(\Omega) = \{a, b, d\}$$
$$\lambda(\Gamma^*(\Omega)) = \frac{4}{6}$$

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\Gamma^*(A)$</th>
<th>$\Gamma_*(A)$</th>
<th>$P^*(A)$</th>
<th>$P_*(A)$</th>
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<tbody>
<tr>
<td>$\emptyset$</td>
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<tr>
<td>${1}$</td>
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<td>${a}$</td>
<td>$\frac{1}{4}$</td>
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<tr>
<td>${2}$</td>
<td>${b, d}$</td>
<td>${b}$</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{1}{4}$</td>
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<tr>
<td>${3}$</td>
<td>${d}$</td>
<td>$\emptyset$</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
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<tr>
<td>${1, 2}$</td>
<td>${a, b, d}$</td>
<td>${a, b}$</td>
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<td>${1, 3}$</td>
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<td>${2, 3}$</td>
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<td>${1, 2, 3}$</td>
<td>${a, b, d}$</td>
<td>${a, b, d}$</td>
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One can consider $P^*(A)$ and $P_*(A)$ as upper and lower probability bounds.
Some properties of probability bounds:

a) $P^*: 2^\Omega \rightarrow [0, 1]$

b) $0 \leq P_* \leq P^* \leq 1$, \hspace{1em} $P_*(\emptyset) = P^*(\emptyset) = 0$, \hspace{1em} $P_*(\Omega) = P^*(\Omega) = 1$

c) $A \subseteq B \Rightarrow P^*(A) \leq P^*(B)$ and $P_*(A) \leq P_*(B)$

d) $A \cap B = \emptyset \Leftrightarrow P^*(A) + P^*(B) = P^*(A \cup B)$

e) $P^*(A \cup B) \geq P_*(A) + P_*(B) - P_*(A \cap B)$

f) $P^*(A \cup B) \leq P^*(A) + P^*(B) - P^*(A \cap B)$

g) $P_*(A) = 1 - P^*(\Omega \setminus A)$
One can prove the following generalized equation:

\[
P_\ast \left( \bigcup_{i=1}^{n} A_i \right) \geq \sum_{\emptyset \neq I : I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot P_\ast \left( \bigcap_{i \in I} A_i \right)
\]

These set functions also play an important role in theoretical physics (capacities, Choquet, 1955). Shafer did generalize these thoughts and developed a theory of belief functions.
How is new knowledge incorporated?

Every observer announces the location of the ship in form of a subset of all possible ship locations. Given these set-valued mappings, we can derive upper and lower probabilities with the help of the observer importance measure. Let us assume the ship is certainly at sea.

How do the upper/lower probabilities change?
a) *Geometric Conditioning*  
(observers that give partial or full wrong information are discarded)

\[
P_*(A | B) = \frac{\lambda(\{\xi \in \Theta | \Gamma(\xi) \subseteq A \text{ and } \Gamma(\xi) \subseteq B\})}{\lambda(\{\xi \in \Theta | \Gamma(\xi) \subseteq B\})} = \frac{P_*(A \cap B)}{P_*(B)}
\]

\[
P^*(A | B) = \frac{\lambda(\{\xi \in \Theta | \Gamma(\xi) \subseteq B \text{ and } \Gamma(\xi) \cap A \neq \emptyset\})}{\lambda(\{\xi \in \Theta | \Gamma(\xi) \subseteq B\})} = \frac{P^*(A \cup \overline{B}) - P^*(\overline{B})}{1 - P^*(\overline{B})}
\]
b) *Data Revision*  
(the observed data is modified such that they fit the certain information)

\[
(P_*)_B(A) = \frac{P_*(A \cup \overline{B}) - P_*(\overline{B})}{1 - P_*(B)}
\]

\[
(P^*)_B(A) = \frac{P^*(A \cap B)}{P^*(B)}
\]

These two concepts have different semantics. There are several more belief revision concepts.
Let \((\Omega, 2^{\Omega})\) be a space of events. Further be \((O_1, 2^{O_1}, \lambda_1)\) and \((O_2, 2^{O_2}, \lambda_2)\) spaces of independent observers.

We call \((O_1 \times O_2, \lambda_1 \cdot \lambda_2)\) the product space of observers and

\[
\Gamma : O_1 \times O_2 \rightarrow 2^\Omega, \Gamma(x_1, x_2) = \Gamma_1(x_1) \cap \Gamma_2(x_2)
\]

the combined observer function.

We obtain with

\[
(P_L)_*(A) = \frac{(\lambda_1 \cdot \lambda_2)(\{(x_1, x_2) \mid \Gamma(x_1, x_2) \neq \emptyset \land \Gamma(x_1, x_2) \subseteq A\})}{(\lambda_1 \cdot \lambda_2)(\{(x_1, x_2) \mid \Gamma(x_1, x_2) \neq \emptyset\})}
\]

the lower probability of \(A\) that respects both observations.
\[ \Omega = \{1, 2, 3\} \]
\[ \lambda_1: \{a\} \mapsto \frac{1}{3} \quad \lambda_2: \{c\} \mapsto \frac{1}{2} \]
\[ \{b\} \mapsto \frac{2}{3} \quad \lambda_2: \{d\} \mapsto \frac{1}{2} \]
\[ O_1 = \{a, b\} \quad \Gamma_1: \begin{align*} a & \mapsto \{1, 2\} \\ b & \mapsto \{2, 3\} \end{align*} \]
\[ O_2 = \{c, d\} \quad \Gamma_2: \begin{align*} c & \mapsto \{1\} \\ d & \mapsto \{2, 3\} \end{align*} \]

Combination:

\[ O_1 \times O_2 = \{\overline{ac}, \overline{bc}, \overline{ad}, \overline{bd}\} \]

\[ \lambda: \begin{align*} \{\overline{ac}\} & \mapsto \frac{1}{6} \\ \{\overline{ad}\} & \mapsto \frac{1}{6} \\ \{\overline{bc}\} & \mapsto \frac{2}{6} \\ \{\overline{bd}\} & \mapsto \frac{2}{6} \end{align*} \]
\[ \Gamma: \begin{align*} \overline{ac} & \mapsto \{1\} \\ \overline{ad} & \mapsto \{2\} \\ \overline{bc} & \mapsto \emptyset \\ \overline{bd} & \mapsto \{2, 3\} \end{align*} \]

\[ \Gamma_*(\Omega) = \{(x_1, x_2) \mid \Gamma(x_1, x_2) \neq \emptyset\} = \{\overline{ac}, \overline{ad}, \overline{bd}\} \]

\[ \lambda(\Gamma_*(\Omega)) = \frac{4}{6} \]
Example (2)

<table>
<thead>
<tr>
<th>$A$</th>
<th>$(P_*)_{\Gamma_1}(A)$</th>
<th>$(P_*)_{\Gamma_2}(A)$</th>
<th>$(P_*)_{\Gamma}(A)$</th>
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</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
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<td>${1, 2}$</td>
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<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
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</tr>
<tr>
<td>${1, 2, 3}$</td>
<td>1</td>
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Belief Functions

Motivation

$(\Theta, Q)$ Sensors

$\Omega$ possible results, $\Gamma : \Theta \rightarrow 2^\Omega$

$P_* : A \mapsto \sum_{B : B \subseteq A} m(B)$ Lower probability (Belief)

$P^* : A \mapsto \sum_{B : B \cap A \neq \emptyset} m(B)$ Upper probability (Plausibility)

$m : A \mapsto Q(\{\theta \in \Theta \mid \Gamma(\theta) = A\})$ mass distribution

Random sets: Dempster (1968)

Belief functions: Shafer (1974)

Development of a completely new uncertainty calculus as an alternative to Probability Theory
Belief Functions (2)

The function $\text{Bel} : 2^\Omega \rightarrow [0, 1]$ is called belief function, if it possesses the following properties:

$$\text{Bel}(\emptyset) = 0$$
$$\text{Bel}(\Omega) = 1$$
$$\forall n \in \mathbb{N} : \forall A_1, \ldots, A_n \in 2^\Omega :$$
$$\text{Bel}(A_1 \cup \cdots \cup A_n) \geq \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (1 - |I|) \cdot \text{Bel}(\bigcap_{i \in I} A_i)$$

If $\text{Bel}$ is a belief function then for $m : 2^\Omega \rightarrow \mathbb{R}$ with
$$m(A) = \sum_{B : B \subseteq A} (-1)^{|A \setminus B|} \cdot \text{Bel}(B)$$
the following properties hold:

$$0 \leq m(A) \leq 1$$
$$m(\emptyset) = 0$$
$$\sum_{A \subseteq \Omega} m(A) = 1$$
Belief Functions (3)

Let $|\Omega| < \infty$ and $f, g : 2^\Omega \rightarrow [0, 1]$.

\[
\forall A \subseteq \Omega: (f(A) = \sum_{B:B \subseteq A} g(B))
\]

$\iff$

\[
\forall A \subseteq \Omega: (g(A) = \sum_{B:B \subseteq A} (-1)^{|A\setminus B|} \cdot f(B))
\]

($g$ is called the Möbius transformed of $f$)

The mapping $m : 2^\Omega \rightarrow [0, 1]$ is called a mass distribution, if the following properties hold:

\[
m(\emptyset) = 0
\]

\[
\sum_{A \subseteq \Omega} m(A) = 1
\]
Example

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\emptyset$</th>
<th>${1}$</th>
<th>${2}$</th>
<th>${3}$</th>
<th>${1, 2}$</th>
<th>${2, 3}$</th>
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<th>${1, 2, 3}$</th>
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<td>$m(A)$</td>
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<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{2}{4}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\text{Bel}(A)$</td>
<td>$0$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$0$</td>
<td>$\frac{2}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{3}{4}$</td>
<td>$1$</td>
</tr>
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</table>

Belief $\equiv$ lower probability with modified semantic

\[
\text{Bel}(\{1, 3\}) = m(\emptyset) + m(\{1\}) + m(\{3\}) + m(\{1, 3\})
\]

\[
m(\{1, 3\}) = \text{Bel}(\{1, 3\}) - \text{Bel}(\{1\}) - \text{Bel}(\{3\})
\]

$m(A)$ measure of the trust/belief that exactly $A$ occurs

$\text{Bel}_m(A)$ measure of total belief that $A$ occurs

$\text{Pl}_m(A)$ measure of not being able to disprove $A$ (plausibility)

\[
\text{Pl}_m(A) = \sum_{B: A \cap B \neq \emptyset} m(B) = 1 - \text{Bel}(\overline{A})
\]

Given one of $m, \text{Bel}$ or $\text{Pl}$, the other two can be efficiently computed.
Knowledge Representation

\[
m(\Omega) = 1, \ m(A) = 0 \text{ else total ignorance}
\]
\[
m(\{\omega_0\}) = 1, \ m(A) = 0 \text{ else value } (\omega_0) \text{ known}
\]
\[
m(\{\omega_i\}) = p_i, \sum_{i=1}^{n} p_i = 1 \text{ Bayesian analysis}
\]

Further kinds of partial ignorance can be modeled.
Belief Revision

Data Revision:
- Mass of $A$ flows onto $A \cap B$.
- Masses are normalized to 1 ($\emptyset$-mass is destroyed)

Geometric Conditioning:
- Masses that do not lie completely inside $B$, flow off
- Normalize

The mass flow can be described by specialization matrices
Motivation: Combination of $m_1$ and $m_2$

$m_1(A_i) \cdot m_2(B_j)$: Mass attached to $A_i \cap B_j$, if only $A_i$ or $B_j$ are concerned

$\sum_{i,j:A_i \cap B_j=A} m_1(A_i) \cdot m_2(B_j)$: Mass attached to $A$ (after combination)

This consideration only leads to a mass distribution, if $\sum_{i,j:A_i \cap B_j=\emptyset} m_1(A_i) \cdot m_2(B_j) = 0$.

If this sum is $> 0$ normalization takes place.
If $m_1$ and $m_2$ are mass distributions over $\Omega$ with belief functions $\text{Bel}_1$ and $\text{Bel}_2$ and does further hold $\sum_{i,j: A_i \cap B_j = \emptyset} m_1(A_i) \cdot m_2(B_j) < 1$, then the function $m : 2^\Omega \to [0, 1], m(\emptyset) = 0$

$$m(A) = \frac{\sum_{B,C: B \cap C = A} m_1(B) \cdot m_2(C)}{1 - \sum_{B,C: B \cap C = \emptyset} m_1(B) \cdot m_2(C)}$$

is a mass distribution. The belief function of $m$ is denoted as $\text{comb}(\text{Bel}_1, \text{Bel}_2)$ or $\text{Bel}_1 \oplus \text{Bel}_2$. The above formula is called the combination rule.
Example

\[ m_1(\{1, 2\}) = \frac{1}{3} \]
\[ m_1(\{2, 3\}) = \frac{2}{3} \]
\[ m_2(\{1\}) = \frac{1}{2} \]
\[ m_2(\{2, 3\}) = \frac{1}{2} \]

\[ m = m_1 \oplus m_2 : \]

\[ \{1\} \mapsto \frac{1}{6} = \frac{1}{4} \]
\[ \{2\} \mapsto \frac{1}{6} = \frac{1}{4} \]
\[ \emptyset \mapsto 0 \]
\[ \{2, 3\} \mapsto \frac{2}{6} = \frac{1}{2} \]
Remarks:

a) The result from the combination rule and the analysis of random sets is identical
b) There are more efficient ways of combination
c) $\text{Bel}_1 \oplus \text{Bel}_2 = \text{Bel}_2 \oplus \text{Bel}_1$
d) $\oplus$ is associative
e) $\text{Bel}_1 \oplus \text{Bel}_1 \neq \text{Bel}_1$ (in general)
f) $\text{Bel}_2 : 2^\Omega \rightarrow [0, 1], m_2(B) = 1$

$$\text{Bel}_2(A) = \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

The combination of $\text{Bel}_1$ and $\text{Bel}_2$ yields the data revision of $m_1$ with $B$. 
The **pignistic transformation** \( Bet \) transforms a normalized mass function \( m \) into a probability measure \( P_m = Bet(m) \) as follows:

\[
P_m(A) = \sum_{\emptyset \neq B \subseteq \Omega} m(B) \frac{|A \cap B|}{|B|}, \forall A \subseteq \Omega.
\]

It can be shown that

\[
bel(A) \leq P_m(A) \leq pl(A)
\]
There are three possible murders

Let $m(\{John\}) = 0.48$, $m(\{John, Mary\}) = 0.12$, 
$m(\{Peter, John\}) = 0.32$, $m(\Omega) = 0.08$

We have:

$$P_m(\{John\}) = 0.48 + \frac{0.12}{2} + \frac{0.32}{2} + \frac{0.08}{3} \approx 0.73$$

$$P_m(\{Peter\}) = \frac{0.32}{2} + \frac{0.08}{3} \approx 0.19$$

$$P_m(\{Mary\}) = \frac{0.12}{2} + \frac{0.08}{3} \approx 0.09$$

The picmistic transformation gives a reasonable "Ranking"
Imprecise Probabilities

Let $x_0$ be the true value but assume there is no information about $P(A)$ to decide whether $x_0 \in A$. There are only probability boundaries.

Let $\mathcal{L}$ be a set of probability measures. Then we call

$$(P_{\mathcal{L}})_*: 2^\Omega \rightarrow [0, 1], A \mapsto \inf \{P(A) \mid P \in \mathcal{L}\}$$

the lower and

$$(P_{\mathcal{L}})^*: 2^\Omega \rightarrow [0, 1], A \mapsto \sup \{P(A) \mid P \in \mathcal{L}\}$$

the upper probability of $A$ w.r.t. $\mathcal{L}$.

a) $(P_{\mathcal{L}})_*(\emptyset) = (P_{\mathcal{L}})^*(\emptyset) = 0$; $(P_{\mathcal{L}})_*(\Omega) = (P_{\mathcal{L}})^*(\Omega) = 1$

b) $0 \leq (P_{\mathcal{L}})_*(A) \leq (P_{\mathcal{L}})^*(A) \leq 1$

c) $(P_{\mathcal{L}})^*(A) = 1 - (P_{\mathcal{L}})_*(\overline{A})$

d) $(P_{\mathcal{L}})_*(A) + (P_{\mathcal{L}})_*(B) \leq (P_{\mathcal{L}})^*(A \cup B)$

e) $(P_{\mathcal{L}})_*(A \cap B) + (P_{\mathcal{L}})_*(A \cup B) \preceq (P_{\mathcal{L}})^*(A) + (P_{\mathcal{L}})^*(B)$
Belief Revision

Let $B \subseteq \Omega$ and $\mathcal{L}$ a class of probabilities. The we call

$$A \subseteq \Omega : (P_{\mathcal{L}})_*(A \mid B) = \inf\{P(A \mid B) \mid P \in \mathcal{L} \land P(B) > 0\}$$

the lower and

$$A \subseteq \Omega : (P_{\mathcal{L}})^*(A \mid B) = \sup\{P(A \mid B) \mid P \in \mathcal{L} \land P(B) > 0\}$$

the upper conditional probability of $A$ given $B$.

A class $\mathcal{L}$ of probability measures on $\Omega = \{\omega_1, \ldots, \omega_n\}$ is of type 1, iff there exist functions $R_1$ and $R_2$ from $2^\Omega$ into $[0, 1]$ with:

$$\mathcal{L} = \{P \mid \forall A \subseteq \Omega : R_1(A) \leq P(A) \leq R_2(A)\}$$
Belief Revision (2)

Intuition: $P$ is determined by $P(\{\omega_i\})$, $i = 1, \ldots, n$ which corresponds to a point in $\mathbb{R}^n$ with coordinates $(P(\{\omega_1\}), \ldots, P(\{\omega_n\}))$.

If $\mathcal{L}$ is type 1, it holds true that:

$$\mathcal{L} \iff \left\{(r_1, \ldots, r_n) \in \mathbb{R}^n \mid \exists P: \forall A \subseteq \Omega:\right.$$  

$$\left. (P_L)^*(A) \leq P(A) \leq (P_L)^*(A) \right.$$  

$$\text{and } r_i = P(\{\omega_i\}), \ i = 1, \ldots, n \right\}$$
Example

\[ \Omega = \{ \omega_1, \omega_2, \omega_3 \} \]
\[ \mathcal{L} = \{ P \ | \ \frac{1}{2} \leq P(\{\omega_1, \omega_2\}) \leq 1, \ \frac{1}{2} \leq P(\{\omega_2, \omega_3\}) \leq 1, \ \frac{1}{2} \leq P(\{\omega_1, \omega_3\}) \leq 1 \} \]

General restriction:
\[ 0 \leq P(\{\omega_i\}) \leq 1 \]
\[ P(\{\omega_1\}) + P(\{\omega_2\}) + P(\{\omega_3\}) = 1 \]

Let \( A_1 = \{ \omega_1, \omega_2 \} \), \( A_2 = \{ \omega_2, \omega_3 \} \), \( A_3 = \{ \omega_1, \omega_3 \} \)

\[ P_*(A_1) + P_*(A_2) + P_*(A_3) - P_*(A_1 \cap A_2) - P_*(A_2 \cap A_3) - P_*(A_1 \cap A_3) + P_*(A_1 \cap A_2 \cap A_3) \]
\[ = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - 0 - 0 - 0 + 0 = \frac{3}{2} > 1 = P(A_1 \cup A_2 \cup A_3) \]
Belief Revision (3)

If $\mathcal{L}$ is type 1 and $(P_{\mathcal{L}})^*(A \cup B) \geq (P_{\mathcal{L}})^*(A) + (P_{\mathcal{L}})^*(B) - (P_{\mathcal{L}})^*(A \cap B)$, then

$$(P_{\mathcal{L}})^*(A \mid B) = \frac{(P_{\mathcal{L}})^*(A \cap B)}{(P_{\mathcal{L}})^*(A \cap B) + (P_{\mathcal{L}})^*(B \cap \overline{A})}$$

and

$$(P_{\mathcal{L}})^*(A \mid B) = \frac{(P_{\mathcal{L}})^*(A \cap B)}{(P_{\mathcal{L}})^*(A \cap B) + (P_{\mathcal{L}})^*(B \cap \overline{A})}$$

Let $\mathcal{L}$ be a class of type 1. $\mathcal{L}$ is of type 2, iff

$$(P_{\mathcal{L}})^*(A_1 \cup \cdots \cup A_n) \geq \sum_{I: \emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \cdot (P_{\mathcal{L}})^*(\bigcap_{i \in I} A_i)$$