Assignment Sheet 4

Assignment 12  Fuzzy Set Operations

Let the following two fuzzy sets be given:

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (8,0) node[anchor=north] {\footnotesize 1 2 3 4 5 6 7 8};
\draw[->] (0,0) -- (0,1) node[anchor=east] {\footnotesize 0 1};
\draw (1,0) -- (1,1); \draw (2,0) -- (2,1); \draw (3,0) -- (3,1); \draw (4,0) -- (4,1); \draw (5,0) -- (5,1); \draw (6,0) -- (6,1); \draw (7,0) -- (7,1); \draw (8,0) -- (8,1);
\draw (1,0.5) -- (4,0.5) node[anchor=north] {$\mu_1$}; \draw (5,0.5) -- (8,0.5) node[anchor=north] {$\mu_2$};
\end{tikzpicture}
\end{center}

Compute and draw for each of the pairs

a) the complement of $\mu_1$ w.r.t. $U = [1,8]$ using the standard fuzzy negation,
b) the intersection of $\mu_1$ and $\mu_2$ using the standard fuzzy $t$-norm $T_{\text{min}},$
c) the intersection of $\mu_1$ and $\mu_2$ using the algebraic product $T_{\text{prod}},$
d) the intersection of $\mu_1$ and $\mu_2$ using the Łukasiewicz $t$-norm $T_{\text{Luka}},$
e) the union of $\mu_1$ and $\mu_2$ using the standard fuzzy $t$-conorm $\perp_{\text{max}},$
f) the union of $\mu_1$ and $\mu_2$ using the algebraic sum $\perp_{\text{sum}},$
g) the union of $\mu_1$ and $\mu_2$ using the Łukasiewicz $t$-conorm $\perp_{\text{Luka}}.$

Assignment 13  Fuzzy Negation

In order to construct an involutive negation, one can use either a strictly monotonously increasing or decreasing generator function:

**Theorem:** $\sim : [0,1] \mapsto [0,1]$ is an involutive fuzzy negation if there exists a continuous function $g : [0,1] \mapsto \mathbb{R}$ that fulfills the following properties:

(i) $g(0) = 0.$

(ii) $g$ is strictly monotonously increasing.

(iii) $\sim a = g^{-1}(g(1) - g(a)).$

**Theorem:** $\sim : [0,1] \mapsto [0,1]$ is an involutive fuzzy negation if there exists a continuous function $f : [0,1] \mapsto \mathbb{R}$ that fulfills the following properties:

(i) $f(1) = 0.$

(ii) $f$ is strictly monotonously decreasing.

(iii) $\sim a = f^{-1}(f(0) - f(a)).$
Now, consider the class of increasing generator functions

\[ g_\lambda(a) = \frac{a}{\lambda + (1 - \lambda)a} \]

Apply the given theorem, which allows to construct an involutive fuzzy negation from an arbitrary continuous and strictly increasing function \( g \) with \( g(0) = 0 \). Draw the resulting fuzzy negation for several values of \( \lambda \).

**Assignment 14  Greatest \( t \)-norm**

Motivate graphically that the Minimum is the greatest \( t \)-norm.
Draw a 3D-Plot for two fuzzy truth variables in [0,1] and the corresponding output variable in [0,1] as e.g. done on slide 8 of the lecture on fuzzy logic.
Start drawing the values necessary for fulfilling the crisp logic, then iteratively add the properties of \( t \)-norms and their graphical meanings in your drawing.

**Assignment 15  Fuzzy Conjunction**

Prove the following theorem which was given in the lecture:

**Theorem:** For all \( t \)-norms \( \top \) and all fuzzy truth values \( a, b \in [0,1] \) it is

\[ \top_{-1}(a, b) \leq \top(a, b) \leq \top_{\min}(a, b), \]

where \( \top_{\min}(a, b) = \min\{a, b\} \) is the standard fuzzy conjunction and \( \top_{-1} \) is the so-called drastic product

\[ \top_{-1}(a, b) = \begin{cases} 
  a & \text{if } b = 1, \\
  b & \text{if } a = 1, \\
  0 & \text{otherwise.} 
\end{cases} \]