OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Fuzzy Data Analysis

Possibilistic Networks

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A Simple Example

Oil contamination of water by trading vessels

Typical formulation:

"The accident occurred 10 miles away from the coast."

Locations of interest: *open sea* (z3), *12-mile zone* (z2), *3-mile zone* (z1), *canal* (ca), *refueling dock* (rd), *loading dock* (ld)

These 6 locations Ω are disjoint and exhaustive

$$\Omega = \{z3, z2, z1, ca, rd, ld\}$$

Modeling Degrees of Belief

Statements are often not simply true or false. Decision maker should be able to quantify their „degree of belief". This can be an objective measurement or subjective valuation. The standard way to model such situations with uncertainty is to use probability theory:

Sample space Θ (finite set of distinct possible outcomes of some random experiment), Events of interest are subsets $A \subseteq \Theta$

The degrees of belief $P : 2^{\Theta} \rightarrow [0, 1]$ are required to satisfy the Kolmogorov axiom

There are good arguments for this choice, e.g. the Dutch Book argument

Kolmogorov Axioms

For **finite** Θ, probability function $P : 2^{\Theta} \rightarrow [0, 1]$ must satisfy

i) $0 \leq P(A) \leq 1$ for all events $A \subseteq \Theta$,

ii) $P(\Theta) = 1$,

iii) if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$ for all A,B

Simple Example

Consider the subjective statement: „The ship is in ca or rd or ld with degree of certainty 0.6, that‘s all I know“

A modelling with probility theory forces the user to specify the degrees of belief for all elementary events. In this case the experts does not want to that. Equal probabilities for rd and ld and equal probabilities for the remaining options are often used. But this is a very precise option that doesn‘t reflect the state of the knowledge (namely ignorance) of the expert.

An alternative solution is to assign beliefs directly to subsets and not to elements, i.e to use mass distributions.

Mass Distribution

Recall example with $\Omega = \{z3, z2, z1, ca, rd, ld\}$

Propositional statement *in port* equals event $\{ca, rd, ld\}$

Event may represent maximum level of differentiation for expert

Expert specifies **mass distribution** $m : 2^{\Omega} \rightarrow [0, 1]$

Here, Ω is called **frame of discernment**

$m : 2^{\Omega} \rightarrow [0, 1]$ must satisfy

(i) $m(\emptyset) = 0$,

(ii) $\sum_{A:A\subseteq\Omega}(A) = 1$

Subsets $A \subseteq \Omega$ with $m(A) > 0$ are called **focal elements** of m

Belief and Plausibility

$m(A)$ measures belief committed *exactly* to A

For *total* amount of belief (**credibility**) of A, sum up $m(B)$ whereas $B \subseteq A$

For *maximum* amount of belief movable to A, sum up $m(B)$ with $B \cap A \neq \emptyset$ **(plausibility)**

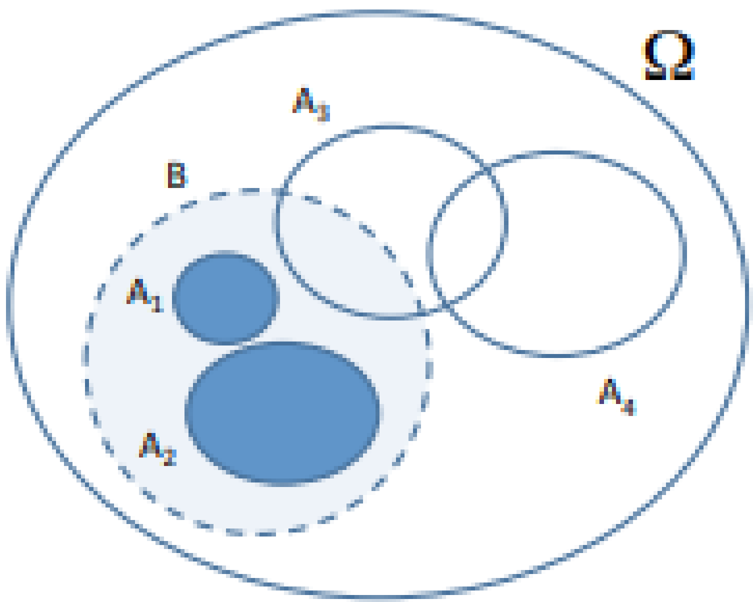
This leads to **belief function** and **plausibility function**

$$\mathrm{Bel}_m : \quad 2^{\Omega} \rightarrow [0, 1], \quad \mathrm{Bel}_m(A) \;=\; \sum_{B : B \subseteq A} m(B)$$

$$\mathrm{Pl}_m : \quad 2^{\Omega} \rightarrow [0, 1], \quad \mathrm{Pl}_m(A) \;=\; \sum_{B : B \cap A \neq \emptyset} m(B)$$

Belief and Plausibility

If the evidence tells us that the truth is in A, and $A \subseteq B$, we say that the evidence supports B.



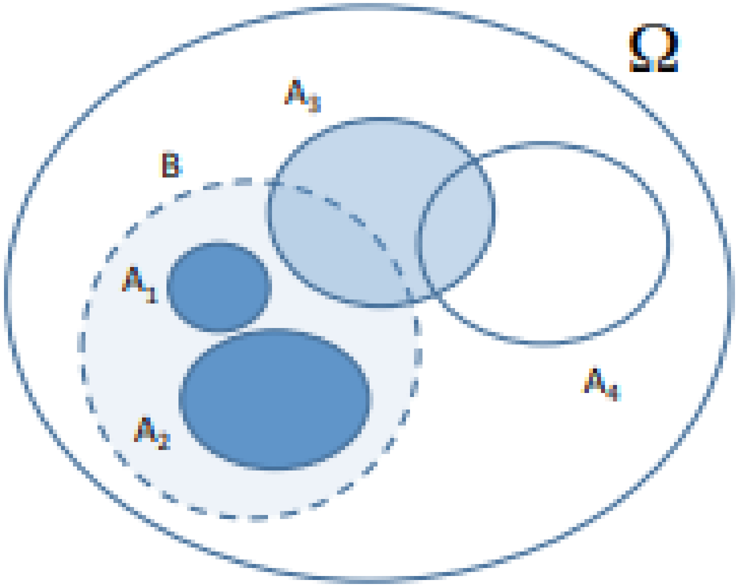
- Given a normalized mass function m, the probability that the evidence supports B is thus

$$Bel(B) = \sum_{A \subseteq B} m(A)$$

- The number $Bel(B)$ is called the degree of belief in B, and the function $B \to Bel(B)$ is called a belief function.

Belief and Plausibility

If the evidence does not support $\overline{B}$, it is consistent with B.



- The probability that the evidence is consistent with B is thus

$$
\begin{aligned}
Pl(B) &= \sum_{A \cap B \neq \emptyset} m(A) \\
&= 1 - Bel(\overline{B}).
\end{aligned}
$$

- The number $Pl(B)$ is called the plausibility of B, and the function $B \to Pl(B)$ is called a plausibility function.

Example

Consider statement: "ship is *in port* with degree of certainty of 0.6, further evidence is not available"

Mass distribution

$m : 2^\Omega \rightarrow [0, 1], m(\{\text{in port}\}) = 0.6, m(\Omega) = 0.4, m(A) = 0$ otherwise

$m(\Omega) = 0.4$ represents inability to attach that amount of mass to any $A \subset \Omega$

e.g. $m(\{\overline{\text{in port}}\}) = 0.4$ would exceed expert's statement

Properties of a Belief Function

Function $Bel : 2^\Omega \to [0,1]$ is a completely monotone capacity: it verifies $Bel(\emptyset) = 0$, $Bel(\Omega) = 1$ and

$$Bel\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right).$$

for any $k \geq 2$ and for any family $A_1, \ldots, A_k$ in 2^Ω.

Conversely, to any completely monotone capacity Bel corresponds a unique mass function m such that:

$$m(A) = \sum_{\emptyset \neq B \subseteq A} (-1)^{|A|-|B|} Bel(B), \quad \forall A \subseteq \Omega.$$

Relations between m, Bel, and PL

Let m be a mass function, Bel and Pl the corresponding belief and plausibility functions

For all $A \subseteq \Omega$,

$$Bel(A) = 1 - Pl(\overline{A})$$

$$m(A) = \sum_{\emptyset \neq B \subseteq A} (-1)^{|A|-|B|} Bel(B)$$

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|+1} Pl(\overline{B})$$

m, Bel and Pl are thus three equivalent representations of

- a piece of evidence or, equivalently
- a state of belief induced by this evidence

Belief and Plausibility

In any case $\mathrm{Bel}(\Omega) = 1$ ("closed world" assumption)

Total ignorance modeled by $m_0 : 2^\Omega \to [0, 1]$ with $m_0(\Omega) = 1$, $m_0(A) = 0$ for all $A \neq \Omega$

m_0 leads to $\mathrm{Bel}(\Omega) = \mathrm{Pl}(\Omega) = 1$ and $\mathrm{Bel}(A) = 0, \mathrm{Pl}(A) = 1$ for all $A \neq \Omega$

For ordinary probability, use $m_1 : 2^\Omega \to [0, 1]$ with $m_1(\{\omega\}) = p_\omega$ and $m_1(A) = 0$ for all sets A with $|A| > 1$

m_1 is called Bayesian belief function

Exact knowledge modeled by $m_2 : 2^\Omega \to [0, 1]$, $m_2(\{\omega_0\}) = 1$ and $m_2(A) = 0$ for all $A \neq \{\omega_0\}$

Possibility Measures can be seen as special belief functions

When the focal sets of m are nested: $A_1 \subset A_2 \subset \ldots \subset A_r$, m is said to be consonant

The following relations then hold

$$Pl(A \cup B) = \max\left(Pl(A), Pl(B)\right), \quad \forall A, B \subseteq \Omega$$

Pl is this a possibility measure, and *Bel* is the dual necessity measure

The possibility distribution is the contour function

$$\pi : \Omega \rightarrow [0, 1], \pi(\omega) = Pl(\{\omega\})$$

The theory of belief function can thus be considered as more expressive than possibility theory (but the combination operations are different, see later).

Possibility and Necessity Measures

$\pi : \Omega \rightarrow [0, 1], \pi(\omega) = Pl(\{\omega\})$

Thus, **possibility measure** and **necessity measure** are defined by

$$\mathrm{poss}_m : 2^{\Omega} \rightarrow [0,1], \quad \Pi_m(B) = \max\{\pi(\omega) : \omega \in B\}$$

$$\mathrm{nec}_m : 2^{\Omega} \rightarrow [0,1], \quad \mathrm{nec}_m(B) = 1 - \Pi(\overline{B})$$

Properties of Possibility Measures

i) $\Pi(\emptyset) = 0$

ii) $\Pi(\Omega) = 1$

iii) $\Pi(A \cup B) = \max\{\Pi(A), \Pi(B)\}$ for all $A, B \subseteq \Omega$

Possibility of some set is determined by its "most possible" element

$\text{nec}(\Omega) = 1 - \Pi(\emptyset) = 1$ means closed world assumption:
"necessarily $\omega_0 \in \Omega$" must be true

Total ignorance: $\Pi(B) = 1, \text{nec}(B) = 0$ for all $B \neq \emptyset, B \neq \Omega$

Perfect knowledge: $\Pi(\{\omega\}) = \text{nec}(\{\omega\}) = 0$ for all $\omega \neq \omega_0$ and
$\Pi(\{\omega_0\}) = \text{nec}(\{\omega_0\}) = 1$

Simple Example

Consider ship locations again

Given membership function

$$\pi(z3) = \pi(z2) = 0$$
$$\pi(z1) = \pi(ld) = 0.3$$
$$\pi(ca) = 0.6$$
$$\pi(rd) = 0.1$$

$\Pi(\{z3, z2\}) = 0$ and $\mathrm{nec}(\{z1, ca, rd, ld\}) = 1$

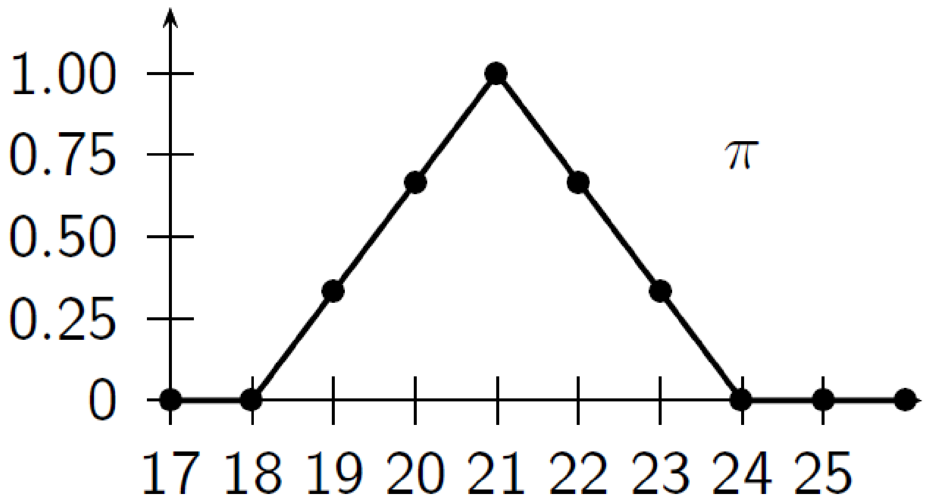
We know it is impossible that ship is located in $\{z3, z2\}$

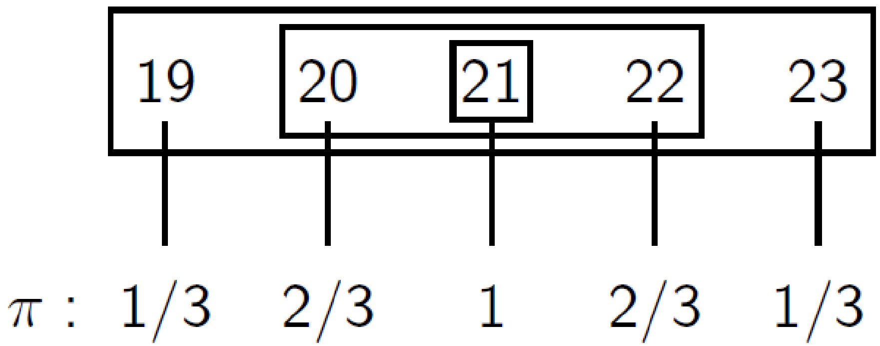
$\Pi(\{ca, rd\}) = 1$, $\mathrm{nec}(\{ca, rd\}) = 0.7$ means “location of ship is possibly but not with certainty in $\{ca, rd\}$”

Possibility and Fuzzy Sets

Let variable T be temperature in $^\circ C$ (only integers)

Current but unknown value T_0 is given by "T is around $21^\circ C$"





Incomplete information induces possibility distribution function π

π is numerically identical with membership function

Nested α-cuts play same role as focal elements

Possibility Theory: An Axiomatic Approach

Definition

Let Ω be a (finite) sample space. A **possibility measure** Π on Ω is a function $\Pi : 2^{\Omega} \to [0, 1]$ satisfying

i) $\Pi(\emptyset) = 0$ and

ii) $\forall E_1, E_2 \subseteq \Omega : \Pi(E_1 \cup E_2) = \max\{\Pi(E_1), \Pi(E_2)\}$.

From axioms, it follows $\Pi(E_1 \cap E_2) \leq \min\{\Pi(E_1), \Pi(E_2)\}$

Attributes are introduced as variables (as in probability theory)

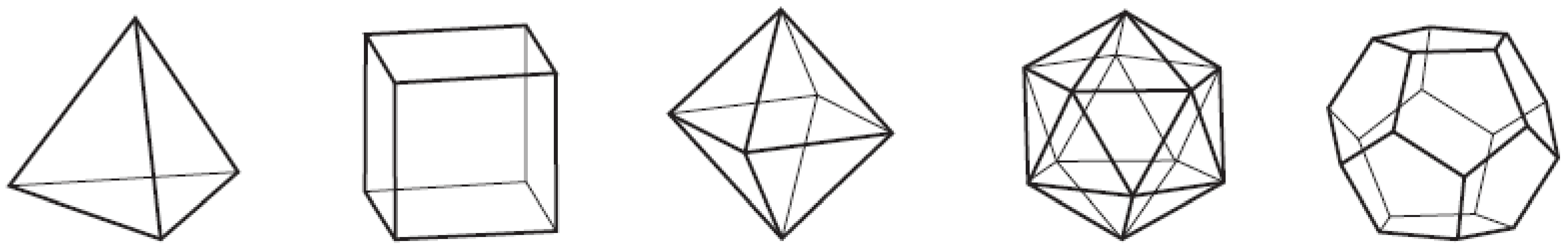
$\Pi(A = a)$ is abbreviation of $\Pi(\{\omega \in \Omega \mid A(\omega) = a\})$

If event E is possible without restriction, then $\Pi(E) = 1$

If event E is impossible, then $\Pi(E) = 0$

Example: Dice and Shakers

shaker 1	shaker 2	shaker 3	shaker 4	shaker 5

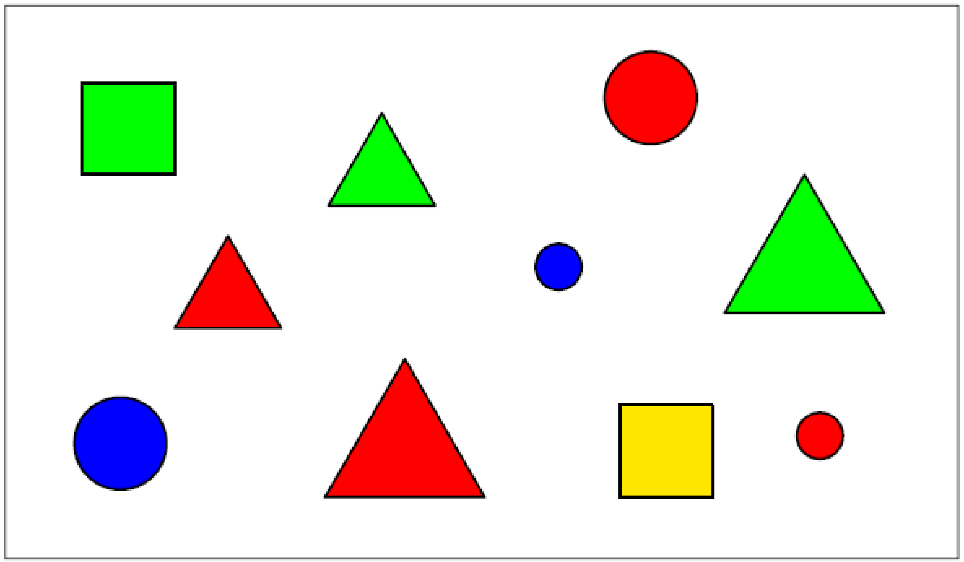


tetrahedron	hexahedron	octahedron	icosahedron	dodecahedron
1 – 4	1 – 6	1 – 8	1 – 10	1 – 12

numbers	degree of possibility
1 – 4	$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5} = 1$
5 – 6	$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5} = \frac{4}{5}$
7 – 8	$\frac{1}{5}+\frac{1}{5}+\frac{1}{5} = \frac{3}{5}$
9 – 10	$\frac{1}{5}+\frac{1}{5} = \frac{2}{5}$
11 – 12	$\frac{1}{5} = \frac{1}{5}$

Example

Example Domain



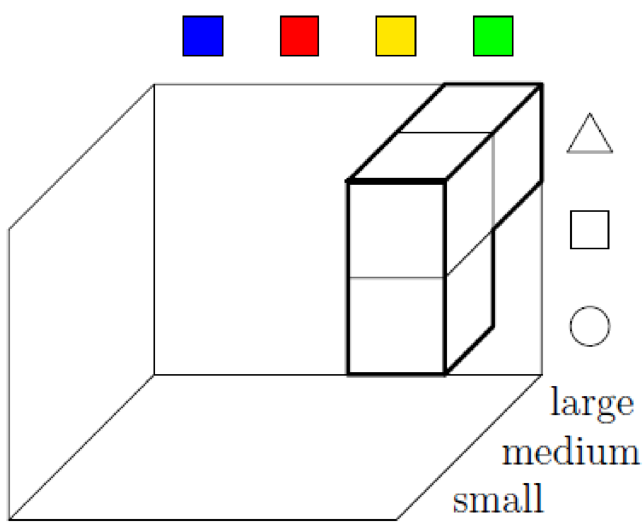
Relation

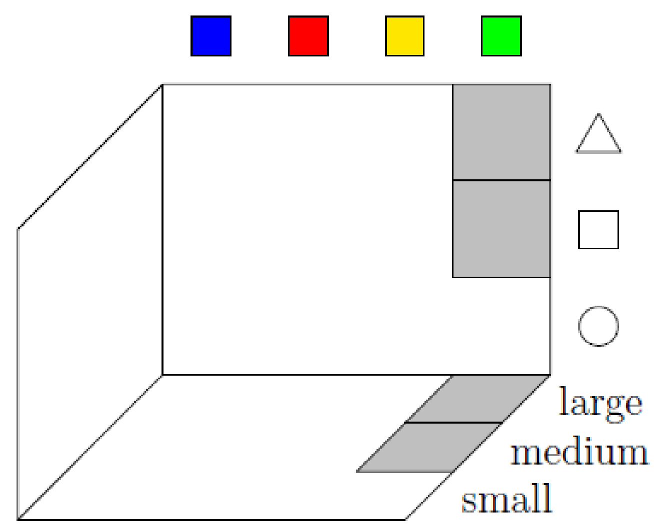
color	shape	size
🟦	○	small
🟦	○	medium
🟥	○	small
🟥	○	medium
🟥	△	medium
🟥	△	large
🟨	□	medium
🟩	□	medium
🟩	△	medium
🟩	△	large

- 10 simple geometrical objects, 3 attributes.
- One object is chosen at random and examined.
- Inferences are drawn about the unobserved attributes.

Example: Reasoning

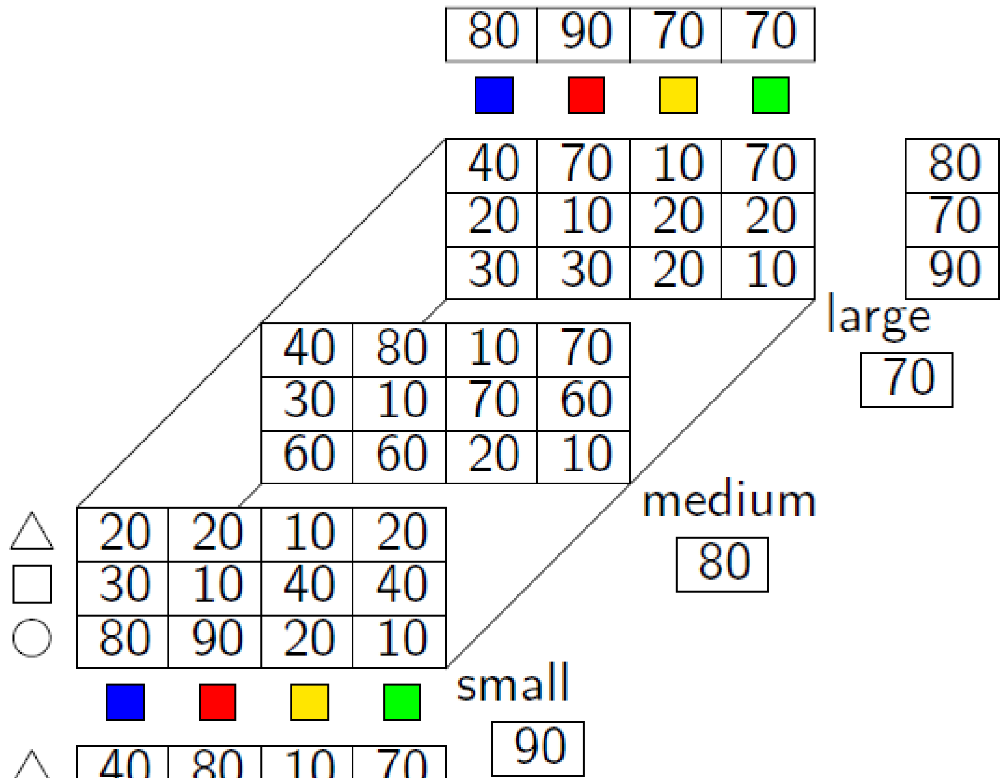
- Let it be known (e.g. from an observation) that the given object is green. This information considerably reduces the space of possible value combinations.
- From the prior knowledge it follows that the given object must be
  - either a triangle or a square and
  - either medium or large.





Example:Possibility Distribution

all numbers in parts per 1000



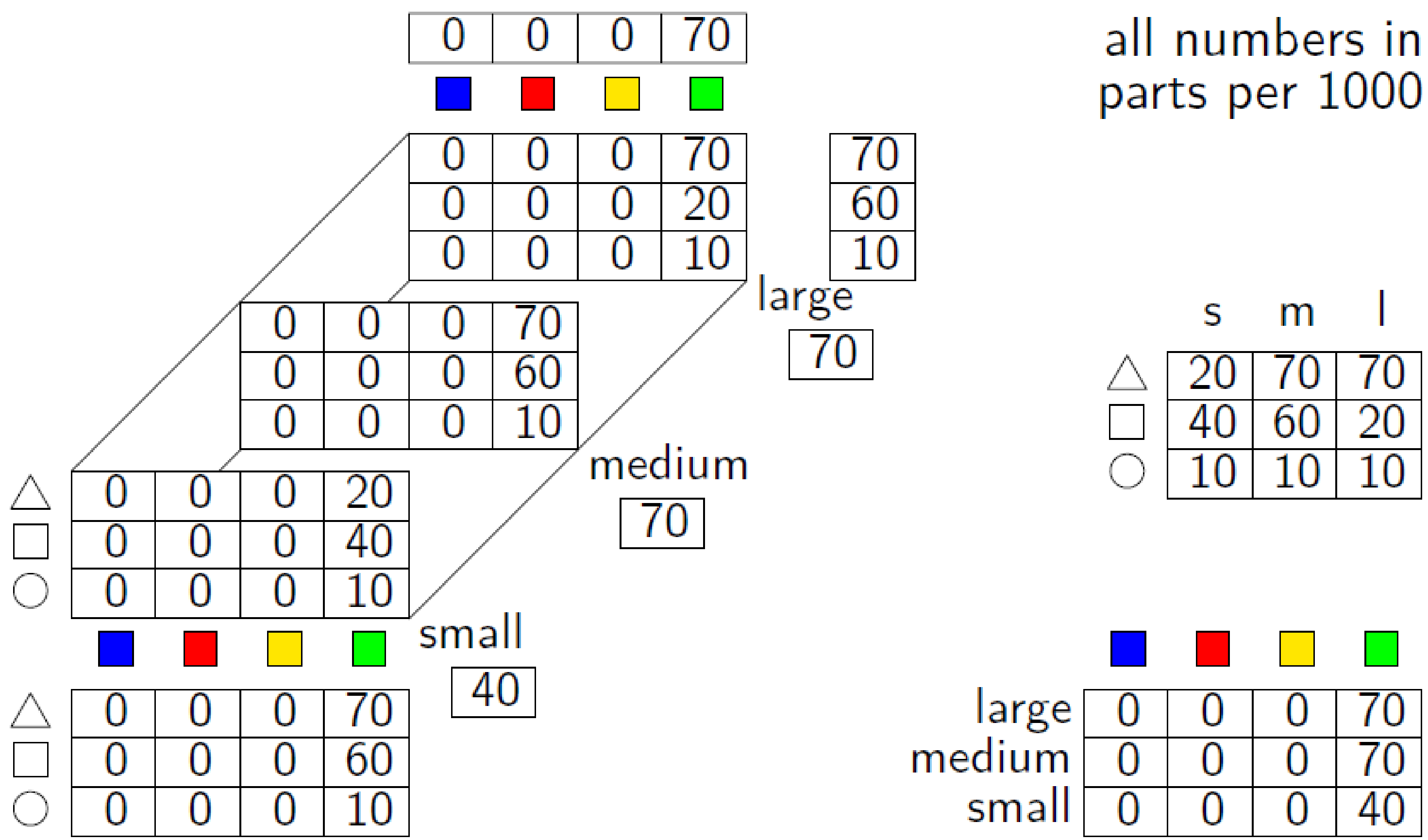
	blue	red	yellow	green
△	40	80	10	70
□	30	10	70	60
○	80	90	20	10

	s	m	l
△	20	80	70
□	40	70	20
○	90	60	30

	blue	red	yellow	green
large	40	70	20	70
medium	60	80	70	70
small	80	90	40	40

Numbers state degrees of possibility of corresponding value combination

Example: Reasoning



From the information, that the object is green, we can derive information obout the possibilities of shape and size. For high dimensional possilities the complexity can be handled by using information about (conditional) independences

Conditional Possibility and Independence

Definition

Let Ω be a (finite) sample space, Π a possibility measure on Ω, and $E_1, E_2 \subseteq \Omega$ events. Then $\Pi(E_1 \mid E_2) = \Pi(E_1 \cap E_2)$ is called the **conditional possibility** of E_1 given E_2.

Definition

Let Ω be a (finite) sample space, Π a possibility measure on Ω, and A, B, and C attributes with respective domains $\text{dom}(A)$, $\text{dom}(B)$, and $\text{dom}(C)$. A and B are called **conditionally possibilistically independent** given C, written $A \perp\!\!\!\perp_\Pi B \mid C$, iff

$$\forall a \in \text{dom}(A) : \forall b \in \text{dom}(B) : \forall c \in \text{dom}(C) :$$
$$\Pi(A = a, B = b \mid C = c) = \min\{\Pi(A = a \mid C = c), \Pi(B = b \mid C = c)\}$$

Possibilistic Networks

Example: Decomposition of a 21-dim possibility distribution by using independences between lower dimensional possibility distributions .

The (hyper-) graph visualized the independence structure by separation properties in the graph, and this representation allows efficient reasoning and learning methods in high dimensional problems.