Multi-criteria Decision Making under Performance and Preference Uncertainty

Mickaël Binois, Jürgen Branke, Alexander Engau, Carlos M. Fonseca,
Salvatore Greco, Miłosz Kadziński, Kathrin Klamroth,
Sanaz Mostaghim, Patrick Reed and Roman Słowiński

March 29, 2018

Abstract

We propose a novel methodology for interactive multi-objective optimization taking into account imprecision, ill-determination and uncertainty referring to both, the technical aspects determining evaluations of solutions by objective functions and the subjective aspects related to the preferences of the decision maker. With this aim, we consider a probability distribution on the space of the objective functions and a probability distribution on the space of the utility functions representing preferences of the decision maker. On the basis of these two probability distributions, without loss of generality supposed to be independent, one can compute a multi-criteria expected utility with a corresponding standard deviation, that permit to assess a quality of each proposed solution. One can also compute an average multi-criteria expected utility and a related standard deviation for a set of solutions, which permit to assess a quality of a population of solutions. This feature can be useful in evolutionary multi-objective optimization algorithms to compare populations of solutions in successive iterations.

1 Introduction

This paper summarizes the work of the *Preference Uncertainty Quantification* working group at the Dagstuhl seminar 18031 "Personalized Multi-objective Programming: An Analytics Perspective" that took place in Schloss Dagstuhl – Leibniz Center for Informatics - on January 14-19, 2018.

2 Uncertainties

When dealing with multi-objective optimization problems, the decision makers (DMs), and the analysts helping them to solve these problems, are confronted in their reasoning with some uncertainties that are inherent to two kinds of "imperfect" information (see [**?**] and [**?**]):

1. Information about the preferences of DMs is always partial and ill-defined. Even more, complete preferences do not exist a priori in DMs' mind, because they evolve in the decision aiding process in interaction with an analyst. The preferences are formed in a constructive learning process in which DMs get a conviction that the most preferred solution has been reached for a given problem statement.

2. Information about consequences of considered solutions usually depend on hardly measurable or random variables. This makes that, in general, the evaluation of solutions with respect to different criteria is imprecise or uncertain.

Therefore, there is a need to take into account these two sources of uncertainty in an interactive multi-objective optimization process. A first consideration of this problem, but taking into account only uncertainty related to utility functions, has been proposed in [**?**].

3 Problem Formulation and Basic Notation

The multi-objective optimization process presented in this paper is formally represented as a multi-objective programming problem under performance and preference uncertainty as follows. Let $X \subset R^n$ be an n-dimensional set of feasible decisions (or solutions, designs, alternatives, etc.) Let $f\colon R^n \to R^m$ be an m-dimensional vector, called objective function, that maps each decision $x \in X$ to a corresponding consequence or performance vector $y = f(x)$. To model performance uncertainty, we assume that each objective function $f = (f_1, f_2, \ldots, f_m)$ is a random element of some (for now: a priori) given set $\mathcal{F}$ of cardinality k, i.e., $\mathcal{F} = \{f^1, f^2, \ldots, f^k\}$ with random outputs $y^i = (y^i_1, y^i_2, \ldots, y^i_m)$ for each $i \in \{1, 2, \ldots, k\}$. In other words, for each $i \in \{1, 2, \ldots, k\}$, the vector function $f^i = (f^i_1, f^i_2, \ldots, f^i_m)$ is one realization of the random objective function f.

Moreover, under the additional assumption that this uncertainty is stochastic in nature, we can assign or estimate a stochastic probability vector $p = (p_1, p_2, \ldots, p_k)$ with $\sum_{i=1}^k p_i = 1$ and with the interpretation that $\Pr[f = f^i] = \Pr[y = y^i] = p_i$ for each $i \in \{1, 2, \ldots, k\}$. In this way, we have defined a discrete probability distribution on the space of values taken by the objective function. Obviously, one can consider a generic probability distribution, not necessarily a discrete one. For a scheme of this setting, see the conceptual relationship between *technical information about the performance* and *conjoint probability distribution on values of the objective function* in Figure 1 on the top.

Similarly, we can describe the uncertainty about preferences of the DM, considering a utility function $u : R^m \to R$, such that $y \mapsto z = u(y)$. Again, u is considered to be an element of a set $\mathcal{U} = \{u^1, \ldots, u^\ell\}$, interpreted as a set of possible realizations of an uncertain utility function. Each utility function $u^j \in \mathcal{U}$ has a probability $\Pr[u = u^j] = q_j$, $j = 1, \ldots, \ell$. This is marked in Figure 1 as *preference information* and *probability distribution of utility function*.

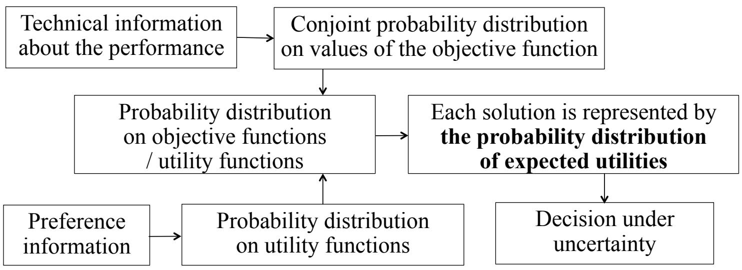

Figure 1: Main idea underlying the proposed methodology

3.1 A Simple Example

Consider a simple example, with $n = 2$ and $X = [0, 1]^2$, so that the decision input to the objective functions is a vector $x = (x_1, x_2)$ composed of two decision variables.

3.1.1 Performance Uncertainty

Let us measure the performance of x in two dimensions, i.e., $m = 2$, so that $f\colon R^2 \to R^2$ with $f = (f_1, f_2)$ for each objective realization. Moreover, consider $k = 3$ uncertain realizations of the objective function, denoted by $\mathcal{F} = \{f^1, f^2, f^3\}$, with probabilities $p = (p_1, p_2, p_3) = (0.5, 0.2, 0.3)$, and taking the following form:

$$
\begin{aligned}
f^1(x) &:= (f^1_1(x_1, x_2), f^1_2(x_1, x_2)) = (x_1, x_2)\\
f^2(x) &:= (f^2_1(x_1, x_2), f^2_2(x_1, x_2)) = (\sqrt{x_1}, \sqrt[3]{x_2})\\
f^3(x) &:= (f^3_1(x_1, x_2), f^3_2(x_1, x_2)) = (x_1^2, x_2^3).
\end{aligned}
$$

Note: Alternatively, supposing that the values taken by the objective function in each realization depend on the value taken on a basic reference realization (for example the mean value in case of an estimation through a Bayesian process) one can define the performance set $Y := \{f(x)\colon x \in X\} \subset R^m$ and then use a transformation $\phi^h\colon R^m \to R^m$ for each possible realization $h = 1, \ldots, k$, so that for each $y = f(x) \in Y$ we can also write $\phi^h(y) = \phi^h(y_1, y_2)$ or $\phi^h(f_1(x), f_2(x)) = (f_1^h(x), f_2^h(x))$. For instance, in the considered example, we can take as a basic reference realization $f^1(x) = f^1(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)) = (y_1, y_2) = (x_1, x_2)$, and for each realization $h = 1, 2, 3$, suppose:

- $\phi^1(y) = \phi^1(y_1, y_2) = (y_1, y_2)$
  or $\phi^1(f(x)) = \phi^1(f_1(x), f_2(x)) = (f_1^1(x), f_2^1(x)) = (f_1(x), f_2(x))$,
- $\phi^2(y) = \phi^2(y_1, y_2) = (\sqrt{y_1}, \sqrt[3]{y_2})$
  or $\phi^2(f(x)) = \phi^2(f_1(x), f_2(x)) = (f_1^2(x), f_2^2(x)) = (\sqrt{f_1(x)}, \sqrt[3]{f_2(x)})$,
- $\phi^3(y) = \phi^3(y_1, y_2) = ((y_1)^2, (y_2)^3)$
  or $\phi^3(f(x)) = \phi^2(f_1(x), f_2(x)) = (f_1^3(x), f_2^3(x)) = ((f_1(x))^2, (f_2(x))^3)$.

3.1.2 Preference Uncertainty

Suppose that we have a probability distribution on a set of $\ell = 4$ utility functions describing the preference information as follows:

$$\begin{aligned} q_1 = 0.4 &\quad : \quad u^1(y) = 0.3y_1 + 0.7y_2, \\ q_2 = 0.3 &\quad : \quad u^2(y) = 0.5y_1 + 0.5y_2, \\ q_3 = 0.2 &\quad : \quad u^3(y) = 0.8y_1 + 0.2y_2, \\ q_4 = 0.1 &\quad : \quad u^4(y) = 0.9y_1 + 0.1y_2, \end{aligned}$$

where q_1, q_2, q_3, q_4 are probabilities of realization of these utility functions.

3.1.3 Expected Utility and Variance of a Single Solution

In the following, we assume that the probability distributions of performance information and utility functions are independent from each other. Therefore, the joint probability distribution on the product space $\mathcal{F} \times \mathcal{U}$ assigns to each pair (f^i, u^j) the probability $\pi_{ij} = p_i \cdot q_j$ shown in the following matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{21} & \pi_{31} \\ \pi_{12} & \pi_{22} & \pi_{32} \\ \pi_{13} & \pi_{23} & \pi_{33} \\ \pi_{14} & \pi_{24} & \pi_{34} \end{bmatrix}^T = \begin{bmatrix} 0.20 & 0.08 & 0.12 \\ 0.15 & 0.06 & 0.09 \\ 0.10 & 0.04 & 0.06 \\ 0.05 & 0.02 & 0.03 \end{bmatrix}^T$$

For each decision x and each realization of its performance f^i in $\mathcal{F}$, one can compute the utility value $u^j(f^i(x))$ that can be presented in the form of a matrix $\mathbf{U}(x)$ with elements $u^j(f^i(x))$ for i and j.

$$\mathbf{U}(x) = \begin{bmatrix} u^1(f^1(x)) & u^2(f^1(x)) & u^3(f^1(x)) & u^4(f^1(x)) \\ u^1(f^2(x)) & u^2(f^2(x)) & u^3(f^2(x)) & u^4(f^2(x)) \\ u^1(f^3(x)) & u^2(f^3(x)) & u^3(f^3(x)) & u^4(f^3(x)) \end{bmatrix}$$

Assuming that $x = (0.5, 0.7)$, one can compute the entries of matrix $\mathbf{U}(x)$, getting:

$$\mathbf{U}(0.5, 0.7) = \begin{bmatrix} 0.6400 & 0.6000 & 0.5400 & 0.5200 \\ 0.8337 & 0.7975 & 0.7433 & 0.7252 \\ 0.3151 & 0.2965 & 0.2686 & 0.2593 \end{bmatrix}$$

In order to compute the expected utility value $E(u(f(x))$ of decision x, we first need to compute the matrix:

$$\mathbf{V}(x) = \mathbf{U}(x) \times \Pi = [\left(u^j(f^i(x) \cdot \pi_{i,j}\right)_{\substack{i=1\ldots,k \\ j=1,\ldots,\ell}}].$$

In our example, we get:

$$\mathbf{V}(0.5, 0.7) = \begin{bmatrix} 0.1280 & 0.0900 & 0.0540 & 0.0260 \\ 0.0667 & 0.0479 & 0.0297 & 0.0145 \\ 0.0378 & 0.0267 & 0.0161 & 0.0078 \end{bmatrix}$$

Then, the expected utility value $E(u(f(x)))$ is obtained as:

$$E(u(f(x))) = \sum_{i=1}^{k}\sum_{j=1}^{\ell} u^j(f^i(x)) \cdot \pi_{ij}. \tag{1}$$

In our example, for $x = (0.5, 0.7)$, the expected utility value is $E(u(f(0.5, 0.7))) = 0.5452$.
The variance is given by:

$$\sigma^2(u(f(x))) = \sum_{i=1}^{k}\sum_{j=1}^{\ell} (u^j(f^i(x)) - E(u(f(x))))^2 \cdot \pi_{ij}, \tag{2}$$

which, in our example, gives $\sigma^2(u(f(x))) = 0.0339$.

In general, the DM will try to maximize the expected value $E(u(f(x)))$ and to minimize the variance of the selected solution $\sigma^2(u(f(x)))$. This principle can be applied in different procedures to select a solution x from a set of feasible solutions $X \in R^n$, such as:

- select a solution $x \in X$ with the maximum expected utility value $E(u(f(x)))$ provided that its variance $\sigma^2(u(f(x)))$ is not greater than a given threshold σ^2*;
- select a solution $x \in X$ with the minimum variance $\sigma^2(u(f(x)))$ provided that its expected utility value is not smaller than a given threshold $E*$;
- select a solution $x \in X$ maximizing a scoring function $S(E(u(f(x))), \sigma^2(u(f(x))))$ being not decreasing with respect to the expected utility value $E(u(f(x)))$ and not increasing with respect to the variance $\sigma^2(u(f(x)))$, as it is the case of

  $$S(E(u(f(x))), \sigma^2(u(f(x)))) = E(u(f(x))) - \lambda \cdot \sigma^2(u(f(x)))$$

  where $\lambda \geq 0$ is a coefficient representing a DM's aversion to risk.

Let us apply the above procedures to a set of feasible solutions $X = \{x^1, x^2, x^3, x^4\}$, where

- $x^1 = (0.5, 0.7)$,
- $x^2 = (0.8, 0.4)$,
- $x^3 = (0.4, 0.8)$,
- $x^4 = (0.9, 0.2)$.

Let us observe that solution x^1 is the same as solution x considered in the above simple example. Computing the expected utility value and the variance for each solution from X we get

- $E(u(f(x^1))) = 0.5452,\ \sigma^2(u(f(x^1))) = 0.0339$,
- $E(u(f(x^2))) = 0.5768,\ \sigma^2(u(f(x^2))) = 0.0350$,
- $E(u(f(x^3))) = 0.5496,\ \sigma^2(u(f(x^3))) = 0.0323$,
- $E(u(f(x^4))) = 0.5643,\ \sigma^2(u(f(x^4))) = 0.0377$.

Consequently:

- if the DM wants to select a solution $x \in X$ with the maximum expected utility value $E(u(f(x)))$ provided that its variance $\sigma^2(u(f(x)))$ is not greater than the threshold $(\sigma^*)^2 = 0.0340$, then solution x^3 is selected;
- if the DM wants to select a solution $x \in X$ with the minimum variance $\sigma^2(u(f(x)))$ provided that its expected utility value is not smaller than the threshold $E^* = 0.55$, then solution x^2 is selected;
- if the DM wants to select a solution $x \in X$ maximizing a scoring function

  $$S(E(u(f(x))), \sigma^2(u(f(x)))) = E(u(f(x))) - 2 \cdot \sigma^2(u(f(x))),$$

  then we get

– $S(E(u(f(x^1))), \sigma^2(u(f(x^1)))) = 0.4773$,
– $S(E(u(f(x^2))), \sigma^2(u(f(x^2)))) = 0.5068$,
– $S(E(u(f(x^3))), \sigma^2(u(f(x^3)))) = 0.4850$,
– $S(E(u(f(x^4))), \sigma^2(u(f(x^4)))) = 0.4890$,

so that solution x^2 is selected.

Another problem that can be considered in this context is the following. Suppose the DM wants to select one solution from $X \subseteq R^n$, which would maximize the expected utility value $E(u(f(x)))$ and minimize the variance $\sigma^2(u(f(x)))$, taking into account a number of constraints concerning decision variables $h_s(x) \leq 0,\ s = 1, \ldots, S$. Formally, this problem can be formulated as follows:

$$\text{maximize: } E(u(f(x)))$$

$$\text{minimize: } \sigma^2(u(f(x)))$$

subject to the constraints

$$x \in X, \tag{3}$$
$$h_s(x) \leq 0,\ s = 1, \ldots, S. \tag{4}$$

Obviously, in general, it is not possible to get an optimum value of $E(u(f(x)))$ and $\sigma^2(u(f(x)))$ for the same feasible x. Instead, one gets a set of Pareto-optimal solutions x, i.e., all solutions $x \in X$ satisfying $h_s(x) \leq 0,\ s = 1, \ldots, S$, for which there does not exist any other solution $\overline{x} \in X$ satisfying $h_s(\overline{x}) \leq 0,\ s = 1, \ldots, S$, having not worse expected utility value $E(u(f(\overline{x})))$ and not worse variance $\sigma^2(u(f(\overline{x})))$, with at least one of the two being better, that is

$$E(u(f(\overline{x}))) > E(u(f(x))), \tag{5}$$
$$\sigma^2(u(f(\overline{x}))) \leq \sigma^2(u(f(x))) \tag{6}$$

or

$$E(u(f(\overline{x}))) \geq E(u(f(x))), \tag{7}$$
$$\sigma^2(u(f(\overline{x}))) < \sigma^2(u(f(x))). \tag{8}$$

Coming back to our example, we have $X = [0, 1]^2$, and let us consider the constraint $h(x) = x_1 + x_2 - 1.25 \leq 0$. Taking into account the set of objective functions $\mathcal{F}$ and the set of utility function $\mathcal{U}$ with respective probability distributions p and q, generating the conjoint probability distribution Π on $\mathcal{F} \times \mathcal{U}$ introduced above, we can get a set of representative Pareto-optimal solutions presented in Table 1.

3.1.4 Expected Utility Value and Variance of a Set of Solutions

Suppose we have a set of solutions $X = \{x^1, \ldots, x^r, \ldots, x^t\} \subseteq R^n$. In this case, it is possible to compute the expected utility value and the variance of this population of solutions, as follows:

$$E(u(f(X)) = \sum_{r=1}^{t} \sum_{i=1}^{k} \sum_{j=1}^{\ell} u^j(f^i(x^r)) \cdot \pi_{ij} \tag{9}$$

$$\sigma^2(u(f(X))) = \sum_{r=1}^{t} \sum_{i=1}^{k} \sum_{j=1}^{\ell} (u_j(f^i(x^r)) - E(u(f(X))))^2 \cdot \pi_{ij} \tag{10}$$

The expected utility value $E(u(f(X)))$ and the variance $\sigma^2(u(f(X)))$ can be computed using expected utility values and variances of particular solutions in the population, as well as covariances between these solutions:

Table 1: A representation of Pareto-optimal solutions

x_1	x_2	Expected value	Variance
0.388	0.862	0.580	0.030
0.351	0.899	0.585	0.031
0.314	0.936	0.592	0.031
0.302	0.948	0.594	0.032
0.292	0.958	0.597	0.032
0.284	0.966	0.598	0.033
0.276	0.974	0.600	0.033
0.270	0.980	0.602	0.034
0.263	0.987	0.603	0.035
0.258	0.992	0.605	0.035
0.252	0.998	0.606	0.036
0.250	1.000	0.607	0.036

$$E(u(f(X))) = \sum_{r=1}^{t} E(u(f(x^r))) \tag{11}$$

$$\sigma^2(u(f(X))) = \sum_{r=1}^{t} \sigma^2(u(f(x^r))) + 2\sum_{r<s} \sigma(u(f(x^r)), u(f(x^s))) \tag{12}$$

where $\sigma(u(f(x^r)), u(f(x^s)))$, $r, s = 1, \ldots, t, r < s$, is the covariance between $u(f(x^r))$ and $u(f(x^s))$, that can be computed as follows:

$$\sigma(u(f(x^r)), u(f(x^s))) = \sum_{i=1}^{k}\sum_{j=1}^{\ell}(u_j(f^i(x^r)) - E(u(f(x^r)))) \cdot (u_j(f^i(x^s)) - E(u(f(x^s)))). \tag{13}$$

The concepts of the expected utility value and the variance of a set of solution can be applied in multi-objective optimization algorithms with a different aim, for example:

- find a subset of solutions $Y \subset X$ of a given cardinality q having the maximum expected utility value $E(u(f(Y)))$, provided that its variance $\sigma^2(u(f(Y)))$ is not greater than a given threshold $\overline{\sigma}^2$; the subset Y can be found by solving the following $0-1$ quadratic programming problem:

$$\text{maximize: } \sum_{r=1}^{t} y_r E(u(f(x^r)))$$

  subject to the constraints

$$\sum_{r=1}^{t} y_r \sigma^2(u(f(x^r))) + 2\sum_{r=1}^{t-1}\sum_{s=r+1}^{t} y_r y_s \sigma(u(f(x^r)), u(f(x^s))) \leq \overline{\sigma}^2, \tag{14}$$

$$\sum_{r=1}^{t} y_r = q, \tag{15}$$

$$y_r \in \{0, 1\},\ r = 1, \ldots, t; \tag{16}$$

  the optimal subset Y will be composed of q solutions $x^r \in X$ with $y_r = 1$;

- find a subset of solutions $Y \subset X$ of a given cardinality q having the minimum variance $\sigma^2(u(f(Y)))$, provided that the its expected value $E(u(f(Y)))$ is not smaller than a given threshold $\overline{E}$; the subset Y can be found by solving the following $0-1$ quadratic programming problem:

$$\text{minimize: } \sum_{r=1}^{t} y_r \sigma^2(u(f(x^r))) + 2\sum_{r=1}^{t-1}\sum_{s=r+1}^{t} y_r y_s \sigma(u(f(x^r)), u(f(x^s)))$$

subject to the constraints

$$\sum_{r=1}^{t} y_r E(u(f(x^r))) \geq \overline{E}, \tag{17}$$

$$\sum_{r=1}^{t} y_r = q, \tag{18}$$

$$y_r \in \{0,1\}, r = 1, \ldots, t; \tag{19}$$

again, the optimal subset Y will be composed of q solutions $x^r \in X$ with $y_r = 1$.

Coming back to our example, let us consider again the solutions from the set $X = \{x^1, x^2, x^3, x^4\}$, and let us compute the covariances $\sigma(u(f(x^r)), u(f(x^s)))$, obtaining the following variance-covariance matrix $\mathbf{\Sigma}(X) = [\sigma(u(f(x^r)), u(f(x^s)))]$, where $\sigma(u(f(x^r)), u(f(x^r))) = \sigma^2(u(f(x^r)))$:

$$\mathbf{\Sigma}(X) = \begin{bmatrix} 0.0339 & 0.0258 & 0.0318 & 0.0157 \\ 0.0258 & 0.0350 & 0.0182 & 0.0334 \\ 0.0318 & 0.0182 & 0.0323 & 0.0064 \\ 0.0157 & 0.0334 & 0.0064 & 0.0377 \end{bmatrix}$$

Let us suppose that the DM wants to select a subset of solutions $Y \subset X$ with cardinality $q = 3$, having the maximum expected utility value $E(u(f(Y)))$. Solving the 0-1 quadratic programming problem presented above, and without considering any constraint on the variance $\sigma^2(u(f(Y)))$, we get that the DM has to select the subset $Y_1 = \{x^2, x^3, x^4\}$ with expected utility value $E(u(f(Y_1))) = 1.6907$ and variance $\sigma^2(u(f(Y_1))) = 0.2211$.

If, in turn, the DM would like to select a subset of solutions $Y \subset X$ with cardinality $q = 3$, having the minimum variance $\sigma^2(u(f(Y)))$, then, by solving the corresponding 0-1 quadratic programming problem presented above, and without considering any constraint on the expected value $E(u(f(Y)))$, the DM would get the subset $Y_2 = \{x^1, x^3, x^4\}$ with expected utility value $E(u(f(Y_2))) = 1.6591$ and variance $\sigma^2(u(f(Y_2))) = 0.2118$.

Suppose now that the DM would like to select a subset of solutions $Y \subset X$ with cardinality $q = 2$, having the maximum expected utility value $E(u(f(Y)))$ but under the condition that the variance $\sigma^2(u(f(Y)))$ is not greater than 0.215. In this case, solving the corresponding 0-1 quadratic programming problem, the DM would get the subset $Y_3 = \{x^3, x^4\}$ with expected utility value $E(u(f(Y_3))) = 1.1139$ and variance $\sigma^2(u(f(Y_3))) = 0.0828$.

Finally, suppose that the DM would like to select a subset of solutions $Y \subset X$ with cardinality $q = 2$, having the minimum variance $\sigma^2(u(f(Y)))$ but under the condition that the expected utility value $E(u(f(Y)))$ is not smaller than 1.1. In this case, the DM would get again the subset $Y_4 = \{x^3, x^4\}$.

The above two problems of selecting a subset of solutions of a given cardinality maximizing the expected utility value with a constraint on the variance, or minimizing the variance with a constraint on the expected value, can be interpreted as a discrete version of the Markowitz portfolio selection problem in the context of multi-objective optimization. It is sensible to consider also the classic continuous Markowitz portfolio selection problem which consists in searching for a vector

$$\mathbf{y} = [y_1, \ldots, y_t], \; y_r \geq 0, \; r, \ldots, t, \; \sum_{r=1}^{t} y_r = 1,$$

that maximizes the expected utility value

$$E(u(f(\mathbf{y}))) = \sum_{r=1}^{t} y_r E(u(f(x^r)))$$

subject to the constraint that the variance $\sigma^2(u(f(\mathbf{y})))$ is not greater than a given threshold $\overline{\sigma}^2$, that is

$$\sigma^2(u(f(\mathbf{y}))) = \sum_{r=1}^{t} y_r \sigma^2(u(f(x^r))) + 2\sum_{r=1}^{t-1} \sum_{s=r+1}^{t} y_r y_s \sigma(u(f(x^r)), u(f(x^s))) \leq \overline{\sigma}^2.$$

The classic Markowitz portfolio selection problem can also be formulated as minimization of the variance $\sigma^2(u(f(\mathbf{y})))$ under the constraint that the expected utility value $E(u(f(\mathbf{y})))$ is not smaller than a given threshold $\overline{E}$.

Coming back to our example, let us suppose that the DM wants to compute the vector $\mathbf{y} = [y_1, \ldots, y_4]$ having the maximum expected utility value $E(u(f(\mathbf{y})))$ but under the condition that the variance $\sigma^2(u(f(\mathbf{y})))$ is not greater than 0.o25. In this case, the optimal vector is

$$\mathbf{y^1} = [0 \quad 0.4223 \quad 0.3487 \quad 0.2289],$$

with its corresponding expected utility value $E(u(f(\mathbf{y^1}))) = 0.5644$ and variance $\sigma^2(u(f(\mathbf{y}^1))) = 0.025$.

Instead, if we suppose that the DM wants to compute a vector $\mathbf{y} = [y_1, \ldots, y_4]$ having the minimum variance $\sigma^2(u(f(\mathbf{y})))$ but under the condition that the expected utility value $E(u(f(\mathbf{y})))$ is not smaller than 0.56, then the optimal vector is

$$\mathbf{y^2} = [0 \quad 0.1599 \quad 0.4285 \quad 0.2118]$$

with its corresponding expected utility value $E(u(f(\mathbf{y^2}))) = 0.56$ and variance $\sigma^2(u(f(\mathbf{y}^2))) = 0.0224$.

Let us finally remark, that the value of $y_r, r = 1, \ldots, t$, can be interpreted as a score assigned by a fitness function to the corresponding solution x^r in an evolutionary optimization algorithm, such that the greater the value of y_r the more probably x^r should be selected to generate a new solution.

3.1.5 Heat map visualization of averages and variances

For a visualization of the situation that is described above consider Figure 2. For any two-dimensional input/decision/design/output variable $x = (x_1, x_2)$ in the domain $[0, 1] \times [0, 1]$, we can compute the mean and variance of the $l \cdot k$ (here, $3 \cdot 4 = 12$) entries of the resulting matrix $U(x)$ or $U(y)$. Then, the figure on its left and right side shows the thus computed mean values and variances for variables x or y of a discretized grid on $[0, 1] \times [0, 1]$.

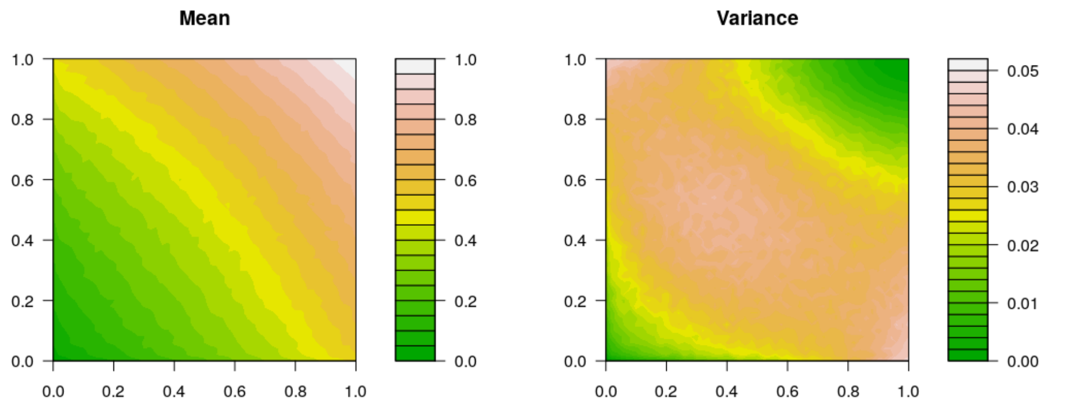

Figure 2: Heat map visualization of averages (on the left) and variances (on the right)

4 Application to Sea-Level Rise and Storm Surge Projections

This section describes a real-world application regarding the deep uncertainties in sea-level rise and storm surge projections. This example represents a probabilistic generalization of the classical Van

Dantzig decision analytical application where the decision is to choose the level of increase in dike height to reduce flood risk [?]. The two objectives are probabilistic as a function of uncertainties in sea level rise due to climate change and local effects of the geophysics of storm surge (i.e., two different but interdependent geophysical models). Figure 3 illustrates the original deterministic Van Dantzig baseline, the mean trade-off between flood risk and investment, as well as the relative locations of the minimum net present values for investment. The challenge as emphasized in the log scale zoomed view is the mean Pareto front would not provide a DM an understanding of the severe variance in the potential outcomes for a given investment. For example, working with the mean trade-off an investment 800 Million US Dollars intended to provide a 1 in 10,000 year level of flood protection has a significant residual probability of dramatically less protection (severe damages and potential loss of life). This probabilistic Pareto space context poses a challenge to decision making, particularly given the potential uncertainties in preferences or risk aversion for the residual risks. It then motivates the question of understanding the potential joint probabilistic outcome of uncertain Pareto performance and uncertain DM's preferences.

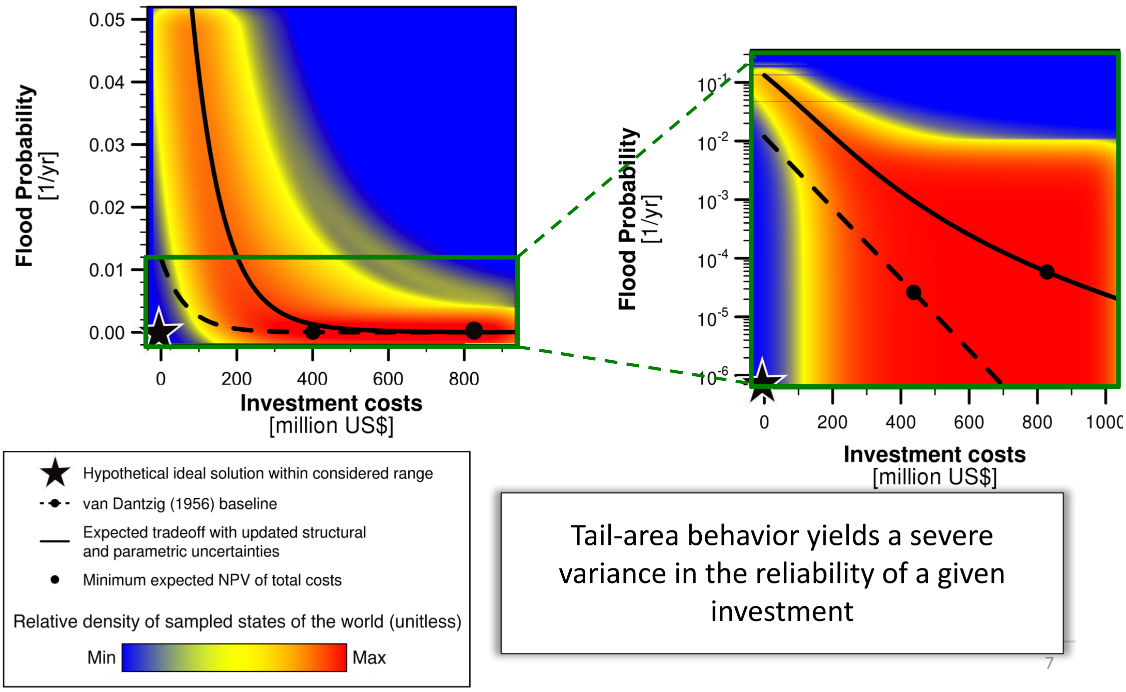

Figure 3: Real-world application about uncertainty in technical information (adapted from [?])

5 Open Questions

In this report, we proposed a novel approach for interactive multi-objective optimization taking into account uncertainty referring to both the evaluations of solutions by objective functions as well as the preferences of the decision maker. We envisage the following directions for future research.

Firstly, we aim at developing methods for elicitation of probability distributions on objective performances and on utility functions. Secondly, we will propose some procedures for robustness analysis that would quantify the stability of results (utilities, ranks, and pairwise relations) obtained in view of uncertain performances and preferences. Thirdly, when aiming to select a set of feasible options, we will account for the interactions between different solutions. Fourthly, we will integrate the proposed methods with evolutionary multi-objective optimization algorithms with the aim of evaluating and selecting a population of solutions. Fifthly, we plan to adapt the introduced approach to a group decision setting, possibly differentiating between two groups of decision makers being responsible for, respectively, setting the goals and compromising these goals based on different utilities. Finally, we will apply the proposed methodology to real-world problems with highly uncertain information about the solutions' performances and decision makers' preferences.