Nonstandard Concepts for Handling Imprecise Data and Imprecise Probabilities

Problems with Probability Theory

Representation of Ignorance

We are given a die with faces $1, \ldots, 6$
What is the certainty of showing up face i ?

- Conduct a statistical survey (roll the die 10000 times) and estimate the relative frequency: $P(\{i\}) = \frac{1}{6}$
- Use subjective probabilities (which is often the normal case): We do not know anything (especially and explicitly we do not have any reason to assign unequal probabilities), so the most plausible distribution is a uniform one.

  Problem: Uniform distribution because of ignorance or extensive statistical tests

Experts analyze aircraft shapes: 3 aircraft types A, B, C
"It is type A or B with 90% certainty. About C, I don't have any clue and I do not want to commit myself. No preferences for A or B."

Problem: Ignorance hard to handle with Bayesian theory

Random Sets: Modeling Imprecise Data

"$A \subseteq X$ being an imprecise date" means: the true value x_0 lies in A but there are no preferences on A.

Ω	set of possible elementary events
$\Theta = \{\xi\}$	set of observers
$\lambda(\xi)$	importance of observer ξ

Some elementary event from Ω occurs and every observer $\xi \in O$ shall announce which elementary events she personally considers possible. This set is denoted by $\Gamma(\xi) \subseteq \Omega$. $\Gamma(\xi)$ is then an imprecise date.

$\lambda : 2^\Theta \to [0,1]$	probability measure
	(interpreted as importance measure)
$(\Theta, 2^\Theta, \lambda)$	probability space
$\Gamma : \Theta \to 2^\Omega$	set-valued mapping

Imprecise Data (2)

Let $A \subseteq \Omega$:

a) $\Gamma^*(A) \stackrel{\text{Def}}{=} \{\xi \in \Theta \mid \Gamma(\xi) \cap A \neq \emptyset\}$

b) $\Gamma_*(A) \stackrel{\text{Def}}{=} \{\xi \in \Theta \mid \Gamma(\xi) \neq \emptyset \text{ and } \Gamma(\xi) \subseteq A\}$

Remarks:

a) If $\xi \in \Gamma^*(A)$, then it is *plausible* for ξ that the occurred elementary event lies in A.

b) If $\xi \in \Gamma_*(A)$, then it is *certain* for ξ that the event lies in A.

c) $\{\xi \mid \Gamma(\xi) \neq \emptyset\} \;=\; \Gamma^*(\Omega) \;=\; \Gamma_*(\Omega)$

Let $\lambda(\Gamma^*(\Omega)) > 0$. Then we call

$$P^*(A) = \frac{\lambda(\Gamma^*(A))}{\lambda(\Gamma^*(\Omega))} \quad \text{the upper, and} \quad P_*(A) = \frac{\lambda(\Gamma_*(A))}{\lambda(\Gamma_*(\Omega))} \quad \text{the lower}$$

probability w. r. t. λ and Γ.

Example

$$\Theta = \{a, b, c, d\} \qquad \lambda: \; a \mapsto {}^1\!/_6 \qquad \Gamma: \; a \mapsto \{1\}$$
$$\Omega = \{1, 2, 3\} \qquad b \mapsto {}^1\!/_6 \qquad b \mapsto \{2\}$$
$$\Gamma^*(\Omega) = \{a, b, d\} \qquad c \mapsto {}^2\!/_6 \qquad c \mapsto \emptyset$$
$$\lambda(\Gamma^*(\Omega)) = {}^4\!/_6 \qquad d \mapsto {}^2\!/_6 \qquad d \mapsto \{2, 3\}$$

A	$\Gamma^*(A)$	$\Gamma_*(A)$	$P^*(A)$	$P_*(A)$
$\emptyset$	$\emptyset$	$\emptyset$	0	0
$\{1\}$	$\{a\}$	$\{a\}$	$\frac{1}{4}$	$\frac{1}{4}$
$\{2\}$	$\{b, d\}$	$\{b\}$	$\frac{3}{4}$	$\frac{1}{4}$
$\{3\}$	$\{d\}$	$\emptyset$	$\frac{1}{2}$	0
$\{1, 2\}$	$\{a, b, d\}$	$\{a, b\}$	1	$\frac{1}{2}$
$\{1, 3\}$	$\{a, d\}$	$\{a\}$	$\frac{3}{4}$	$\frac{1}{4}$
$\{2, 3\}$	$\{b, d\}$	$\{b, d\}$	$\frac{3}{4}$	$\frac{3}{4}$
$\{1, 2, 3\}$	$\{a, b, d\}$	$\{a, b, d\}$	1	1

One can consider $P^*(A)$ and $P_*(A)$ as upper and lower probability bounds.

Imprecise Data (3)

Some properties of probability bounds:

a) $P^*\colon\ 2^\Omega \to [0,1]$

b) $0 \le P_* \le P^* \le 1, \qquad P_*(\emptyset) = P^*(\emptyset) = 0, \qquad P_*(\Omega) = P^*(\Omega) = 1$

c) $A \subseteq B \quad \Rightarrow \quad P^*(A) \le P^*(B)$ and $P_*(A) \le P_*(B)$

d) $A \cap B = \emptyset \quad \not\Rightarrow \quad P^*(A) + P^*(B) = P^*(A \cup B)$

e) $P_*(A \cup B) \ge P_*(A) + P_*(B) - P_*(A \cap B)$

f) $P^*(A \cup B) \le P^*(A) + P^*(B) - P^*(A \cap B)$

g) $P_*(A) = 1 - P^*(\Omega \backslash A)$

Imprecise Data (4)

One can prove the following generalized equation:

$$P_*(\bigcup_{i=1}^{n} A_i) \quad \geq \sum_{\emptyset \neq I : I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot P_*(\bigcap_{i \in I} A_i)$$

These set functions also play an important role in theoretical physics (capacities, Choquet, 1955). Shafer did generalize these thoughts and developed a theory of belief functions.

Belief Revision

How is new knowledge incoporated?

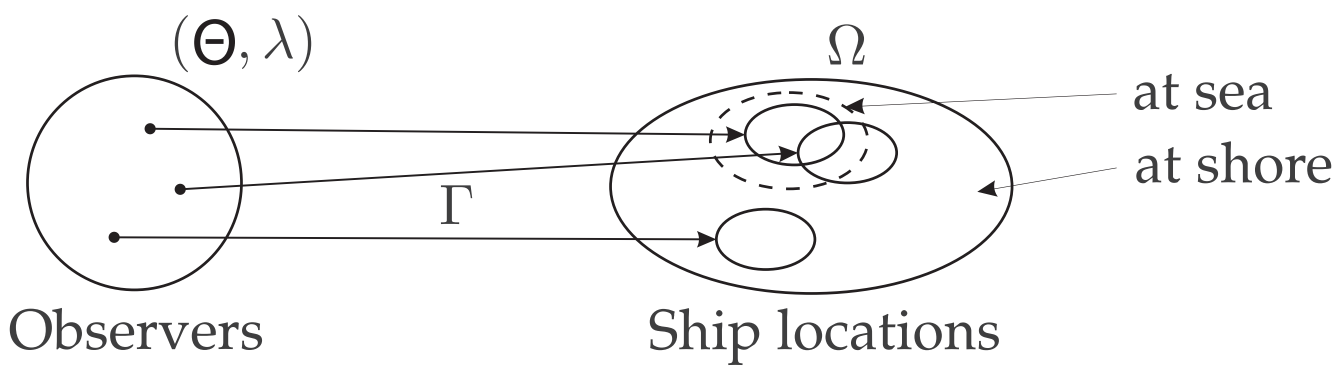
Every observer announces the location of the ship in form of a subset of all possible ship locations. Given these set-valued mappings, we can derive upper and lower probabilities with the help of the observer importance measure. Let us assume the ship is certainly at sea.

How do the upper/lower probabilities change?

a) *Geometric Conditioning*
(observers that give partial or full wrong information are discarded)

$$P_*(A \mid B) = \frac{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq A \text{ and } \Gamma(\xi) \subseteq B\})}{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq B\})} = \frac{P_*(A \cap B)}{P_*(B)}$$

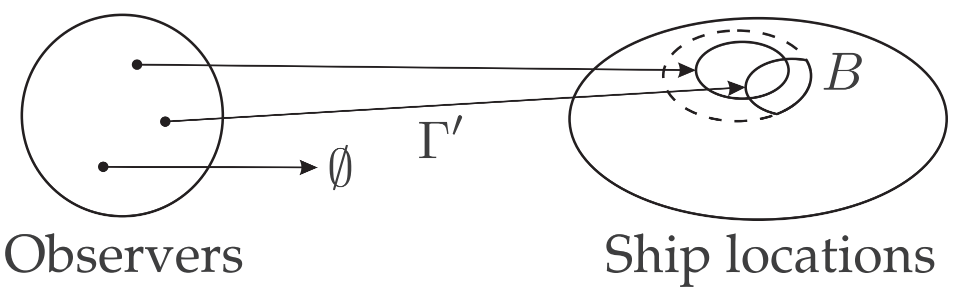
$$P^*(A \mid B) = \frac{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq B \text{ and } \Gamma(\xi) \cap A \neq \emptyset\})}{\lambda(\{\xi \in \Theta \mid \Gamma(\xi) \subseteq B\})} = \frac{P^*(A \cup \overline{B}) - P^*(\overline{B})}{1 - P^*(\overline{B})}$$



b) *Data Revision*
(the observed data is modified such that they fit the certain information)

$$(P_*)_B(A) = \frac{P_*(A \cup \overline{B}) - P_*(\overline{B})}{1 - P_*(B)}$$

$$(P^*)_B(A) = \frac{P^*(A \cap B)}{P^*(B)}$$



These two concepts have different semantics. There are several more belief revision concepts.

Combination of Random Sets

Let $(\Omega, 2^\Omega)$ be a space of events. Further be $(O_1, 2^{O_1}, \lambda_1)$ and $(O_2, 2^{O_2}, \lambda_2)$ spaces of independent observers.

We call $(O_1 \times O_2, \lambda_1 \cdot \lambda_2)$ the product space of observers and

$$\Gamma : O_1 \times O_2 \to 2^\Omega, \Gamma(x_1, x_2) = \Gamma_1(x_1) \cap \Gamma_2(x_2)$$

the combined observer function.

We obtain with

$$(P_L)_*(A) = \frac{(\lambda_1 \cdot \lambda_2)(\{(x_1, x_2) \mid \Gamma(x_1, x_2) \neq \emptyset \wedge \Gamma(x_1, x_2) \sqsubseteq A\})}{(\lambda_1 \cdot \lambda_2)(\{(x_1, x_2 \mid \Gamma(x_1, x_2) \neq \emptyset)\})}$$

the lower probability of A that respects both observations.

Example

$$\Omega = \{1,2,3\} \qquad \lambda_1\colon\ \{a\} \mapsto {}^1\!/_3 \qquad \lambda_2\colon\ \{c\} \mapsto {}^1\!/_2$$

$$\{b\} \mapsto {}^2\!/_3 \qquad \lambda_2\colon\ \{d\} \mapsto {}^1\!/_2$$

$$O_1 = \{a,b\} \qquad \Gamma_1\colon\ a \mapsto \{1,2\} \qquad \Gamma_2\colon\ c \mapsto \{1\}$$

$$O_2 = \{c,d\} \qquad b \mapsto \{2,3\} \qquad d \mapsto \{2,3\}$$

Combination:

$$O_1 \times O_2 \;=\; \{\overline{ac}, \overline{bc}, \overline{ad}, \overline{bd}\}$$

$$\begin{aligned}
\lambda\colon\ \{\overline{ac}\} &\mapsto {}^1\!/_6 & \Gamma\colon\ \overline{ac} &\mapsto \{1\} \\
\{\overline{ad}\} &\mapsto {}^1\!/_6 & \overline{ad} &\mapsto \{2\} \\
\{\overline{bc}\} &\mapsto {}^2\!/_6 & \overline{bc} &\mapsto \emptyset \\
\{\overline{bd}\} &\mapsto {}^2\!/_6 & \overline{bd} &\mapsto \{2,3\}
\end{aligned}$$

$$\begin{aligned}
\Gamma_*(\Omega) &= \{(x_1,x_2) \mid \Gamma(x_1,x_2) \neq \emptyset\} \\
&= \{\overline{ac}, \overline{ad}, \overline{bd}\}
\end{aligned}$$

$$\lambda(\Gamma_*(\Omega)) = {}^4\!/_6$$

Example (2)

A	$(P_*)_{\Gamma_1}(A)$	$(P_*)_{\Gamma_2}(A)$	$(P_*)_{\Gamma}(A)$
$\emptyset$	0	0	0
$\{1\}$	0	$1/2$	$1/4$
$\{2\}$	0	0	$1/4$
$\{3\}$	0	0	0
$\{1, 2\}$	$1/3$	$1/2$	$1/2$
$\{1, 3\}$	0	$1/2$	$1/4$
$\{2, 3\}$	$2/3$	$1/2$	$3/4$
$\{1, 2, 3\}$	1	1	1

Belief Functions

Motivation

(Θ, Q)	Sensors	
Ω	possible results, $\Gamma : \Theta \to 2^{\Omega}$	
$P_* :$	$A \mapsto \sum_{B : B \subseteq A} m(B)$	Lower probability (Belief)
$P^* :$	$A \mapsto \sum_{B : B \cap A \neq \emptyset} m(B)$	Upper probability (Plausibility)
$m :$	$A \mapsto Q(\{\theta \in \Theta \mid \Gamma(\theta) = A\})$	mass distribution

Random sets: Dempster (1968)

Belief functions: Shafer (1974)
Development of a completely new uncertainty calculus as an alternative to Probability Theory

Belief Functions (2)

The function $\mathrm{Bel} : 2^\Omega \to [0,1]$ is called *belief function*, if it possesses the following properties:

$$\mathrm{Bel}(\emptyset) = 0$$

$$\mathrm{Bel}(\Omega) = 1$$

$$\forall n \in \mathbb{N}\colon\ \forall A_1, \dots, A_n \in 2^\Omega :$$
$$\mathrm{Bel}(A_1 \cup \dots \cup A_n) \geq \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \cdot \mathrm{Bel}(\bigcap_{i \in I} A_i)$$

If Bel is a belief function then for $m : 2^\Omega \to \mathbb{R}$ with $m(A) = \sum_{B: B \subseteq A} (-1)^{|A \setminus B|} \cdot \mathrm{Bel}(B)$ the following properties hold:

$$0 \leq m(A) \leq 1$$

$$m(\emptyset) = 0$$

$$\sum_{A \subseteq \Omega} m(A) = 1$$

Belief Functions (3)

Let $|\Omega| < \infty$ and $f, g : 2^\Omega \to [0, 1]$.

$$\forall A \subseteq \Omega \colon \left(f(A) = \sum_{B : B \subseteq A} g(B)\right)$$

$$\Leftrightarrow$$

$$\forall A \subseteq \Omega \colon \left(g(A) = \sum_{B : B \subseteq A} (-1)^{|A \setminus B|} \cdot f(B)\right)$$

(g is called the *Möbius transformed* of f)

The mapping $m : 2^\Omega \to [0, 1]$ is called a *mass distribution*, if the following properties hold:

$m(\emptyset) = 0$

$\sum_{A \subseteq \Omega} m(A) = 1$

Example

A	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2,3\}$
$m(A)$	0	$^1/_4$	$^1/_4$	0	0	0	$^2/_4$	0
$\mathrm{Bel}(A)$	0	$^1/_4$	$^1/_4$	0	$^2/_4$	$^1/_4$	$^3/_4$	1

Belief $\widehat{=}$ lower probability with modified semantic

$$\mathrm{Bel}(\{1,3\}) = m(\emptyset) + m(\{1\}) + m(\{3\}) + m(\{1,3\})$$
$$m(\{1,3\}) = \mathrm{Bel}(\{1,3\}) - \mathrm{Bel}(\{1\}) - \mathrm{Bel}(\{3\})$$

$m(A)$	measure of the trust/belief that exactly A occurs
$\mathrm{Bel}_m(A)$	measure of total belief that A occurs
$\mathrm{Pl}_m(A)$	measure of not being able to disprove A (plausibility)

$$\mathrm{Pl}_m(A) = \sum_{B:A\cap B\neq\emptyset} m(B) = 1 - \mathrm{Bel}(\overline{A})$$

Given one of m, Bel or Pl, the other two can be efficiently computed.

$m(\Omega) = 1$, $m(A) = 0$ else	total ignorance
$m(\{\omega_0\}) = 1$, $m(A) = 0$ else	value (ω_0) known
$m(\{\omega_i\}) = p_i, \sum_{i=1}^{n} p_i = 1$	Bayesian analysis

Further kinds of partial ignorance can be modeled.

Belief Revision

Data Revision:

- Mass of A flows onto $A \cap B$.
- Masses are normalized to 1 ($\emptyset$-mass is destroyed)

Geometric Conditioning:

- Masses that do not lie completely inside B, flow off
- Normalize

The mass flow can be described by specialization matrices

Combinations of Mass Distributions

Motivation: Combination of m_1 and m_2

$m_1(A_i) \cdot m_2(B_j)$:	Mass attached to $A_i \cap B_j$, if only A_i or B_j are concerned
$\sum_{i,j: A_i \cap B_j = A} m_1(A_i) \cdot m_2(B_j)$:	Mass attached to A (after combination)

This consideration only leads to a mass distribution,
if $\sum_{i,j: A_i \cap B_j = \emptyset} m_1(A_i) \cdot m_2(B_j) = 0$.

If this sum is > 0 normalization takes place.

Combination Rule

If m_1 and m_2 are mass distributions over Ω with belief functions Bel_1 and Bel_2 and does further hold $\sum_{i,j:A_i \cap B_j = \emptyset} m_1(A_i) \cdot m_2(B_j) < 1$, then the function $m : 2^\Omega \to [0,1]\,, m(\emptyset) = 0$

$$m(A) = \frac{\sum_{B,C:B \cap C = A} m_1(B) \cdot m_2(C)}{1 - \sum_{B,C:B \cap C = \emptyset} m_1(B) \cdot m_2(C)}$$

is a mass distribution. The belief function of m is denoted as $\text{comb}(\text{Bel}_1, \text{Bel}_2)$ or $\text{Bel}_1 \oplus \text{Bel}_2$. The above formula is called the combination rule.

Example

$$m_1(\{1,2\}) = {}^1\!/_3 \qquad m_2(\{1\}) = {}^1\!/_2$$

$$m_1(\{2,3\}) = {}^2\!/_3 \qquad m_2(\{2,3\}) = {}^1\!/_2$$

$$m = m_1 \oplus m_2 :$$

$$\{1\} \mapsto \frac{{}^1\!/_6}{{}^4\!/_6} = {}^1\!/_4$$

$$\{2\} \mapsto \frac{{}^1\!/_6}{{}^4\!/_6} = {}^1\!/_4$$

$$\emptyset \mapsto 0$$

$$\{2,3\} \mapsto \frac{{}^2\!/_6}{{}^4\!/_6} = {}^1\!/_2$$

Combination Rule (2)

Remarks:

a) The result from the combination rule and the analysis of random sets is identical

b) There are more efficient ways of combination

c) $\text{Bel}_1 \oplus \text{Bel}_2 = \text{Bel}_2 \oplus \text{Bel}_1$

d) $\oplus$ is associative

e) $\text{Bel}_1 \oplus \text{Bel}_1 \neq \text{Bel}_1$ (in general)

f) $\text{Bel}_2 : 2^\Omega \to [0, 1]\,, m_2(B) = 1$

$$\text{Bel}_2(A) = \begin{cases} 1 & \text{if} B \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

The combination of Bel_1 and Bel_2 yields the data revision of m_1 with B.

Decision Making with the Pignistic Transformation

The **pignistic transformation** Bet transforms a normalized mass function m into a probability measure $P_m = Bet(m)$ as follows:

$$P_m(A) = \sum_{\emptyset \neq B \subseteq \Omega} m(B) \frac{|A \cap B|}{|B|}, \forall A \subseteq \Omega.$$

It can be shown that

$$bel(A) \leq P_m(A) \leq pl(A)$$

Decision Making - Example

There are three possible murders

Let $m(\{John\}) = 0.48, \quad m(\{John, Mary\}) = 0.12,$
$m(\{Peter, John\}) = 0.32, \quad m(\Omega) = 0.08$

We have:

$$P_m(\{John\}) = 0.48 + \frac{0.12}{2} + \frac{0.32}{2} + \frac{0.08}{3} \approx 0.73$$

$$P_m(\{Peter\}) = \frac{0.32}{2} + \frac{0.08}{3} \approx 0.19$$

$$P_m(\{Mary\}) = \frac{0.12}{2} + \frac{0.08}{3} \approx 0.09$$

The picmistic transformation givs a reasonable "Ranking"

Imprecise Probabilities

Let x_0 be the true value but assume there is no information about $P(A)$ to decide whether $x_0 \in A$. There are only probability boundaries.

Let $\mathcal{L}$ be a set of probability measures. Then we call

$$(P_{\mathcal{L}})_* : 2^{\Omega} \to [0,1]\,, A \mapsto \inf\{P(A) \mid P \in \mathcal{L}\} \qquad \text{the lower and}$$

$$(P_{\mathcal{L}})^* : 2^{\Omega} \to [0,1]\,, A \mapsto \sup\{P(A) \mid P \in \mathcal{L}\} \qquad \text{the upper}$$

probability of A w. r. t. $\mathcal{L}$.

a) $(P_{\mathcal{L}})_*(\emptyset) = (P_{\mathcal{L}})^*(\emptyset) = 0; \quad (P_{\mathcal{L}})_*(\Omega) = (P_{\mathcal{L}})^*(\Omega) = 1$

b) $0 \leq (P_{\mathcal{L}})_*(A) \leq (P_{\mathcal{L}})^*(A) \leq 1$

c) $(P_{\mathcal{L}})^*(A) = 1 - (P_{\mathcal{L}})_*(\overline{A})$

d) $(P_{\mathcal{L}})_*(A) + (P_{\mathcal{L}})_*(B) \leq (P_{\mathcal{L}})_*(A \cup B)$

e) $(P_{\mathcal{L}})_*(A \cap B) + (P_{\mathcal{L}})_*(A \cup B) \ngeq (P_{\mathcal{L}})_*(A) + (P_{\mathcal{L}})_*(B)$

Belief Revision

Let $B \subseteq \Omega$ and $\mathcal{L}$ a class of probabilities. The we call

$$A \subseteq \Omega : (P_{\mathcal{L}})_*(A \mid B) = \inf\{P(A \mid B) \mid P \in \mathcal{L} \wedge P(B) > 0\} \qquad \text{the lower and}$$
$$A \subseteq \Omega : (P_{\mathcal{L}})^*(A \mid B) = \sup\{P(A \mid B) \mid P \in \mathcal{L} \wedge P(B) > 0\} \qquad \text{the upper}$$

conditional probability of A given B.

A class $\mathcal{L}$ of probability measures on $\Omega = \{\omega_1, \ldots, \omega_n\}$ is of type 1, iff there exist functions R_1 and R_2 from 2^Ω into $[0, 1]$ with:

$$\mathcal{L} = \{P \mid \forall A \subseteq \Omega : R_1(A) \leq P(A) \leq R_2(A)\}$$

Intuition: P is determined by $P(\{\omega_i\})$, $i = 1, \ldots, n$ which corresponds to a point in $\mathbb{R}^n$ with coordinates $\big(P(\{\omega_1\}), \ldots, P(\{\omega_n\})\big)$.

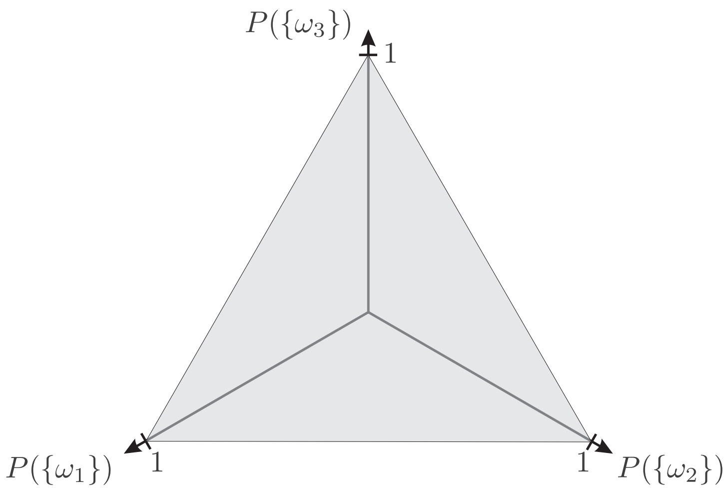
If $\mathcal{L}$ is type 1, it holds true that:

$$\mathcal{L} \Leftrightarrow \Big\{(r_1, \ldots, r_n) \in \mathbb{R}^n \mid \exists P \colon \forall A \subseteq \Omega \colon \\ (P_{\mathcal{L}})_*(A) \leq P(A) \leq (P_{\mathcal{L}})^*(A) \\ \text{and} \;\; r_i = P(\{\omega_i\}),\; i = 1, \ldots, n\Big\}$$

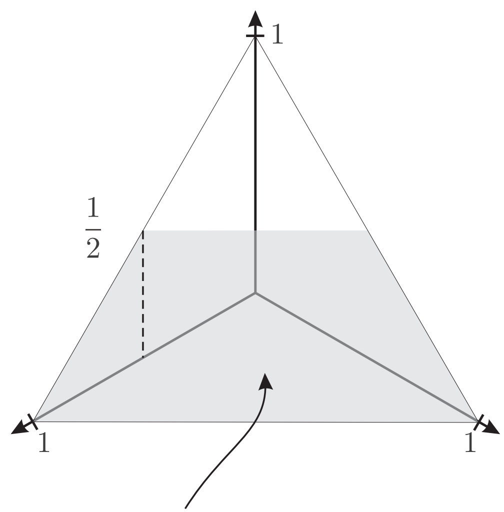
Example

$\Omega = \{\omega_1, \omega_2, \omega_3\}$

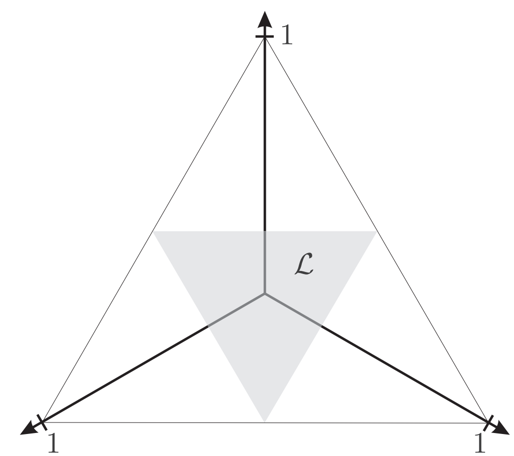
$\mathcal{L} = \{P \mid \frac{1}{2} \leq P(\{\omega_1, \omega_2\}) \leq 1, \quad \frac{1}{2} \leq P(\{\omega_2, \omega_3\}) \leq 1, \quad \frac{1}{2} \leq P(\{\omega_1, \omega_3\}) \leq 1\}$



general restriction:
$0 \leq P(\{\omega_i\}) \leq 1$
$P(\{\omega_1\}) + P(\{\omega_2\}) + P(\{\omega_3\}) = 1$



$\{P \mid \frac{1}{2} \leq P(\{\omega_1, \omega_2\}) \leq 1\}$



Let $A_1 = \{\omega_1, \omega_2\}$, $A_2 = \{\omega_2, \omega_3\}$, $A_3 = \{\omega_1, \omega_3\}$

$$\begin{aligned} & P_*(A_1)+P_*(A_2)+P_*(A_3)-P_*(A_1 \cap A_2)-P_*(A_2 \cap A_3)-P_*(A_1 \cap A_3)+P_*(A_1 \cap A_2 \cap A_3) \\ & = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}-0-0-0+0=\frac{3}{2}>1=P(A_1 \cup A_2 \cup A_3) \end{aligned}$$

Belief Revision (3)

If $\mathcal{L}$ is type 1 and $(P_\mathcal{L})^*(A \cup B) \geq (P_\mathcal{L})^*(A) + (P_\mathcal{L})^*(B) - (P_\mathcal{L})^*(A \cap B)$, then

$$(P_\mathcal{L})^*(A \mid B) = \frac{(P_\mathcal{L})^*(A \cap B)}{(P_\mathcal{L})^*(A \cap B) + (P_\mathcal{L})_*(B \cap \overline{A})}$$

and

$$(P_\mathcal{L})_*(A \mid B) = \frac{(P_\mathcal{L})_*(A \cap B)}{(P_\mathcal{L})_*(A \cap B) + (P_\mathcal{L})^*(B \cap \overline{A})}$$

Let $\mathcal{L}$ be a class of type 1. $\mathcal{L}$ is of type 2, iff

$$(P_\mathcal{L})_*(A_1 \cup \cdots \cup A_n) \geq \sum_{I:\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot (P_\mathcal{L})_*(\bigcap_{i \in I} A_i)$$