Fuzzy Logic

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The Traditional or Aristotlelian Logic

Traditional logic has been founded by Aristotle (384-322 B.C.).

This classical logic can be seen as formal approach to human reasoning.

It's still used today in Artificial Intelligence for knowledge representation and reasoning about knowledge.

Detail of "The School of Athens" by R. Sanzio (1509) showing Plato (left) and his student Aristotle (right).

Propositional Logic

Logics study methods/principles of **reasoning**.

The most famous logic is the **propositional calculus.**

A **proposition** can be (only) *true* or *false,* the calculus uses **connectives** such as „and“ (∧), „or“(∨), „not“(¬), „imply“(→).

The calculus uses **inference rules** (e.g. modus ponens):

Premise 1: It's raining → It's cloudy
Premise 2: It's raining
Conclusion: It's cloudy

The propositional logic based on finite set of logic variables as well as the finite set theory satisfy the axioms of a finite **Boolean algebra**.

A *Boolean algebra* on a set B is defined as quadruple $\mathcal{B} = (B, +, \cdot, \bar{\ })$ where B has at least two elements (bounds) 0 and 1, $+$ and $\cdot$ are binary operators on B, and $\bar{\ }$ is a unary operator on B for which the following properties hold.

Properties of Boolean Algebras I

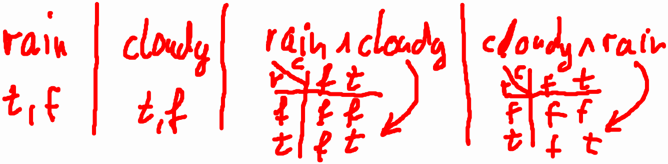
2^X	$\cup$	$\cap$	c
B	$+$	$\cdot$	$-$

(B1) Idempotence	$a + a = a$	$a \cdot a = a$
(B2) Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
(B3) Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(B4) Absorption	$a + (a \cdot b) = a$	$a \cdot (a + b) = a$
(B5) Distributivity	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	$a + (b \cdot c) = (a + b) \cdot (a + c)$
(B6) Universal Bounds	$a + 0 = a,\ a + 1 = 1$	$a \cdot 1 = a,\ a \cdot 0 = 0$
(B7) Complementary	$a + \overline{a} = 1$	$a \cdot \overline{a} = 0$
(B8) Involution	$\overline{\overline{a}} = a$	
(B9) Dualization	$\overline{a + b} = \overline{a} \cdot \overline{b}$	$\overline{a \cdot b} = \overline{a} + \overline{b}$

Properties (B1)-(B4) are common to every **lattice**,

i.e. a Boolean algebra is a distributive (B5), bounded (B6), and complemented (B7)-(B9) lattice,

i.e. every Boolean algebra can be characterized by a partial ordering on a set, *i.e.* $a \leq b$ if $a \cdot b = a$ or, alternatively, if $a + b = b$.



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Boolean Algebras: Set Theory, Propositional Logic

Every theorem in one theory has a counterpart in each other theory.

Counterparts can be obtained applying the following substitutions:

Meaning	Set Theory	Boolean Algebra	Prop. Logic
values	2^X	B	$L(V)$
"meet"/"and"	$\cap$	$\cdot$	$\wedge$
"join"/"or"	$\cup$	$+$	$\vee$
"complement"/"not"	^{c}	$\overline{}$	$\neg$
identity element	X	1	1
zero element	$\emptyset$	0	0
partial order	$\subseteq$	$\leq$	$\rightarrow$

power set 2^X, set of logic variables V, set of all combinations $L(V)$ of truth values of V

The Basic Principle of Classical Logic

The Principle of Bivalence:

"Every proposition is either true or false."

It has been formally developed by Tarski.

Alfred Tarski (1902-1983)

Łukasiewicz suggested to replace it by
The Principle of Valence:

"Every proposition has a truth value."

Propositions can have intermediate truth value, expressed by a number from the unit interval [0, 1].

Jan Łukasiewicz (1878-1956)

How to create create a new formal logic?

- **Formal Language** (Symbols, Operators, Well-formed formulas, Formation rules,..)

- **Truth Functions** and Truth Tables

- **Tautologies** (true for all possible truth-value assignments)

- **Deduction** System (modus ponens, resolution, modus tollens,…)

- **Meta Theoretic Properties** (Completeness, Soundness, Consistency, Truth Functionality)

Fuzzy Logic is a many-valued logic, more than the two truth-values w,f are allowed

Three-valued Logics

A 2-valued logic can be extended to a 3-valued logic *in several ways*, *i.e.* different three-valued logics have been well established:

truth, falsity, indeterminacy are denoted by 1, 0, and 1/2, resp.

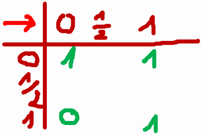
The negation $\neg a$ is defined as $1 - a$, *i.e.* $\neg 1 = 0$, $\neg 0 = 1$ and $\neg 1/2 = 1/2$.

Other primitives, *e.g.* $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, differ from logic to logic.

Five well-known three-valued logics (named after their originators) are defined in the following.

Primitives of Some Three-valued Logics

		Łukasiewicz				Bochvar				Kleene				Heyting				Reichenbach			
a	b	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1



All of them fully conform the usual definitions for $a, b \in \{0, 1\}$.
They differ from each other only in their treatment of $1/2$.
Question: Do they satisfy the law of contradiction ($a \wedge \neg a = 0$) and the law of excluded middle ($a \vee \neg a = 1$)?

n-valued Logics

After the three-valued logics: generalizations to n-valued logics for arbitrary number of truth values $n \geq 2$.

In the 1930s, various n-valued logics were developed.

Usually truth values are assigned by rational number in $[0, 1]$.

Key idea: uniformly divide $[0, 1]$ into n truth values.

Definition
The set T_n of truth values of an n-valued logic is defined as

$$T_n = \left\{0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1\right\}.$$

These values can be interpreted as degree of truth.

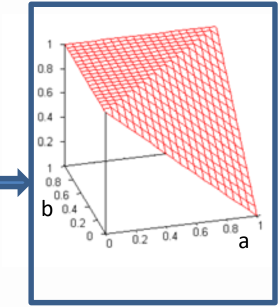
Łukasiewicz Logics

Łukasiewicz proposed a series of *n*-valued logics L_n with truth degrees in T_n

The so called **standard Łukasiewicz logic** has truth degrees in [0, 1]

and uses the following connectives:

$\neg a = 1 - a$	complement
$a \wedge b = \min(a, b)$	weak conjunction
$a . b = \max(0, a + b - 1)$	strong conjunction
$a \vee b = \max(a, b)$	weak disjunction
$a \times b = \min(1, a+b)$	strong disjunction
$a \rightarrow b = \min(1, 1 + b - a)$	implication
$a \leftrightarrow b = 1 - \lvert a - b \rvert$	biimplication



Zadeh‘s „Fuzzy Logic"

In 1965, Zadeh proposed to use the „fuzzy logic“ with values in [0, 1]:

$$
\begin{aligned}
\neg a &:= 1 - a, \\
a \wedge b &:= \min(a, b), \\
a \vee b &:= \max(a, b).
\end{aligned}
$$

Note: In applications the notion of
a „Fuzzy Logic" is often use
in a “broader” sense!