Fuzzy Relations

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A grey level picture interpreted as a fuzzy set

(x_3, y_8)

Y

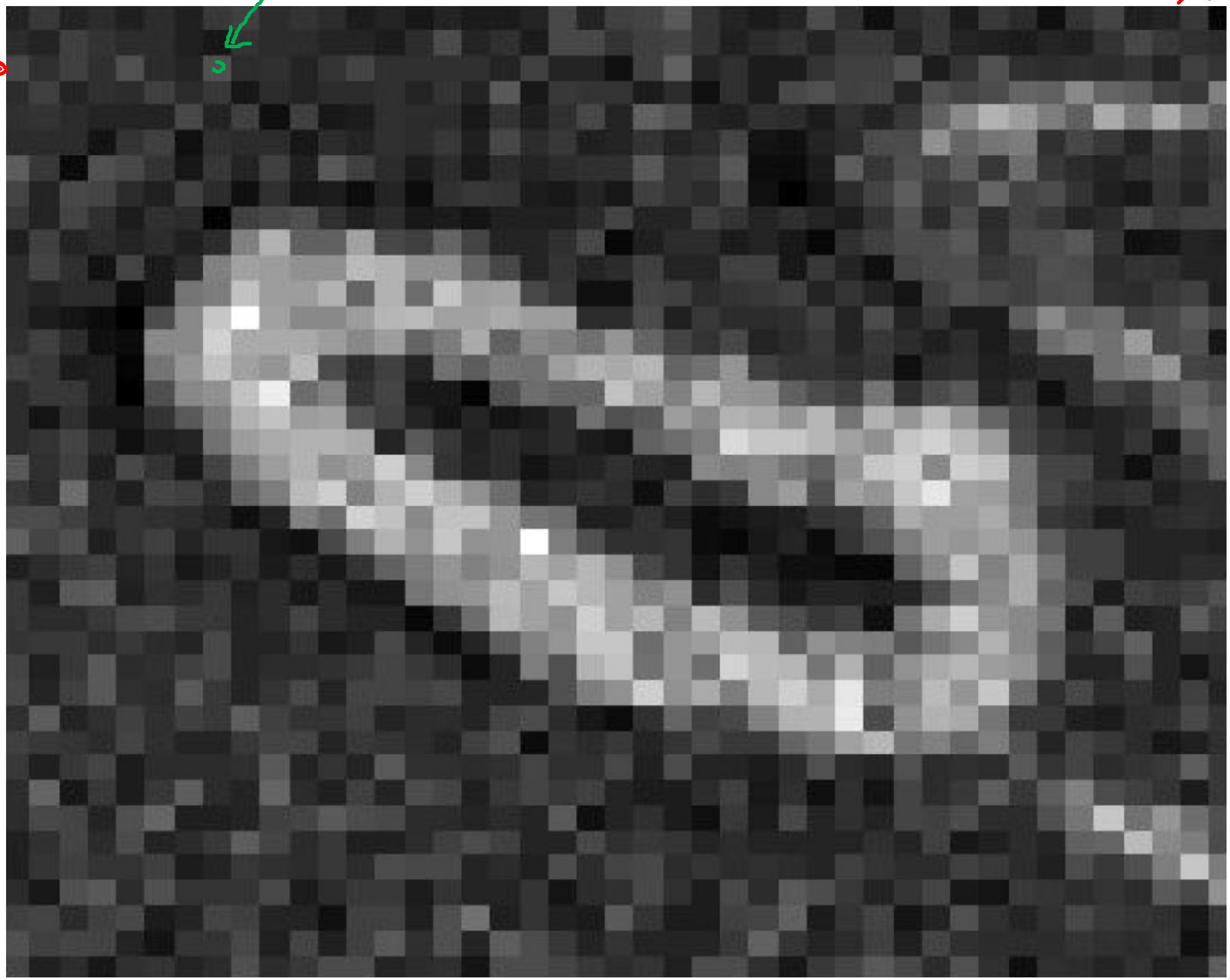
$y_1 \ldots$ y_n

x_1

x_2

X

x_m



$\mu : X \times Y \to [0,1]$

$(x_3, y_8) \mapsto 0.4$

Definition of Relation

A **relation** among crisp sets $X_1, \ldots, X_n$ is a subset of the Cartesian Product $X_1 \times \ldots \times X_n$. It is denoted as $R(X_1, \ldots, X_n)$ or $R(X_i \mid 1 \leq i \leq n)$.

So, the relation $R(X_1, \ldots, X_n) \subseteq X_1 \times \ldots \times X_n$ is set, too. The

basic concept of sets can be also applied to relations:

- containment, subset, union, intersection, complement

Each crisp relation can be defined by its characteristic function

$$R(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if and only if } (x_1, \ldots, x_n) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

The membership of $(x_1, \ldots, x_n)$ in R indicates whether the elements of $(x_1, \ldots, x_n)$ are related to each other or not.

Relation as Ordered Set of Tuples

A relation can be written as a set of ordered tuples.

Thus $R(X_1, \ldots, X_n)$ represents n-dim. membership array $\boldsymbol{R} = [r_{i_1,\ldots,i_n}]$.

- Each element of i_1 of $\boldsymbol{R}$ corresponds to exactly one member of X_1.
- Each element of i_2 of $\boldsymbol{R}$ corresponds to exactly one member of X_2.
- And so on...

If $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ corresponds to $r_{i_1,\ldots,i_n} \in \boldsymbol{R}$, then

$$r_{i_1,\ldots,i_n} = \begin{cases} 1, & \text{if and only if } (x_1, \ldots, x_n) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

Fuzzy Relations

The characteristic function of a crisp relation can be generalized to allow tuples to have degrees of membership.

A **fuzzy relation R** is a fuzzy set of $X_1 \times . . . \times X_n$

The membership grade indicates strength of the present relation between elements of the tuple.

The fuzzy relation can also be represented by an *n*-dimensional membership array.

Example

Let R be a fuzzy relation between two sets X = {New York City, Paris} and Y = {Beijing, New York City, London}.

R shall represent relational concept "very far".

It can be represented (subjectively) as two-dimensional membership array:

	NYC	Paris
Beijing	1	0.9
NYC	0	0.7
London	0.6	0.3

Cartesian Product of Fuzzy Sets: nDimensions

Let $A_1, \ldots, A_n$ be fuzzy sets (n ≥ 2) in $X_1, \ldots, X_n$, respectively

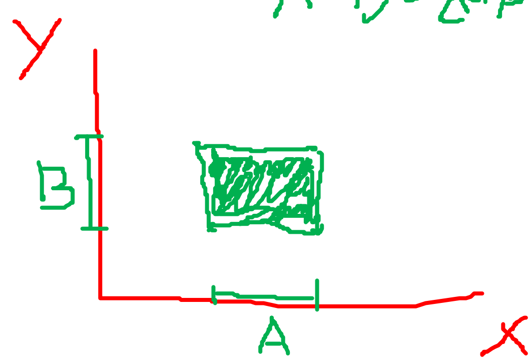
The (fuzzy) *Cartesian product* of $A_1, \ldots, A_n$, denoted by $A_1 \times \ldots \times A_n$, is a fuzzy relation of the product space $X_1 \times \ldots \times X_n$.

It is defined by its membership function

$$\mu_{A_1 \times \ldots \times A_n}(x_1, \ldots, x_n) = T\,(\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n))$$

$$\textit{for } x_i \in X_i,\ 1 \le i \le n.$$

$A \times B = \{(a,b) \mid a \in A, b \in B\}$



In most applications T = min is chosen.

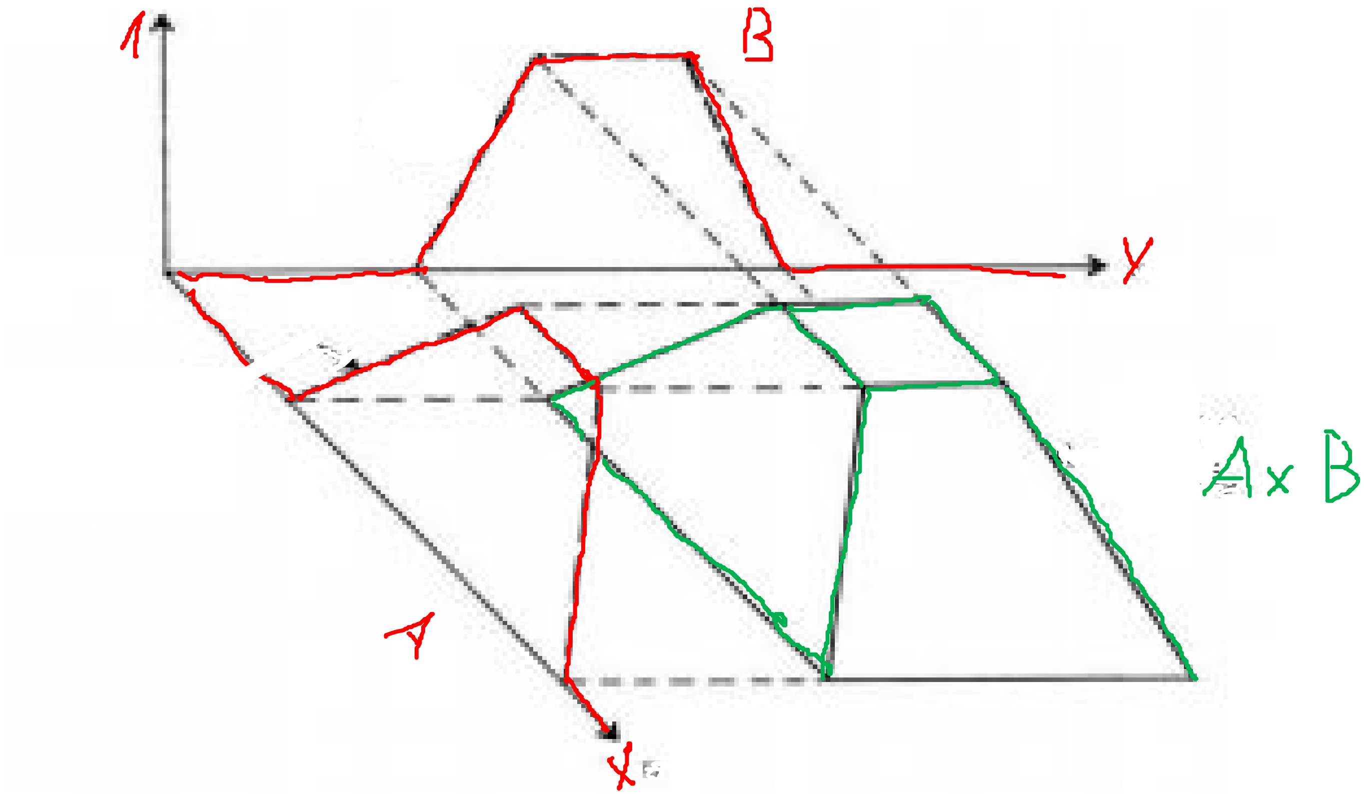
Cartesian Product of Fuzzy Sets in two Dimensions

A special case of the Cartesian product is when $n = 2$.

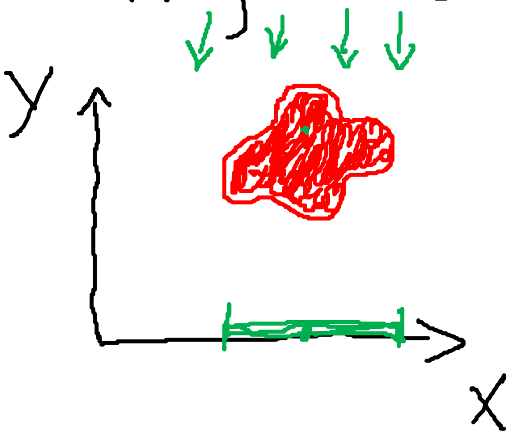
Then the Cartesian product of fuzzy sets $A \in \mathrm{F}(X)$ and $B \in \mathrm{F}(Y)$ is a fuzzy relation $A \times B \in \mathrm{F}(X \times Y)$ defined by

$$\mu_{A \times B}(x, y) = \top\,[\mu_A(x),\ \mu_B(y)],\ \text{for all } x \in X, \text{ and } y \in Y.$$

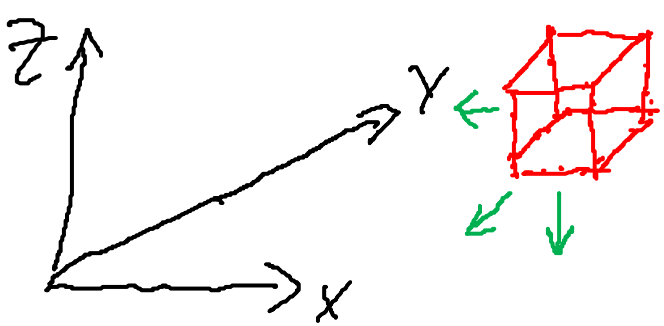
Example: Cartesian Product in F($X \times Y$) with t-norm = min



Projection

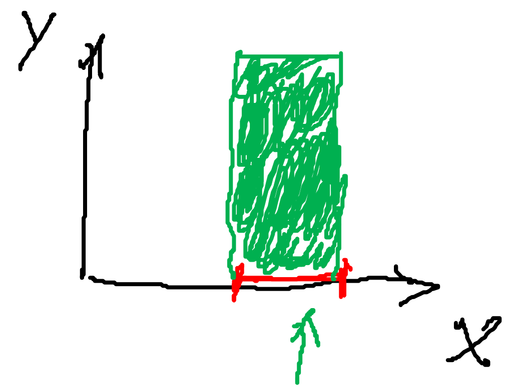


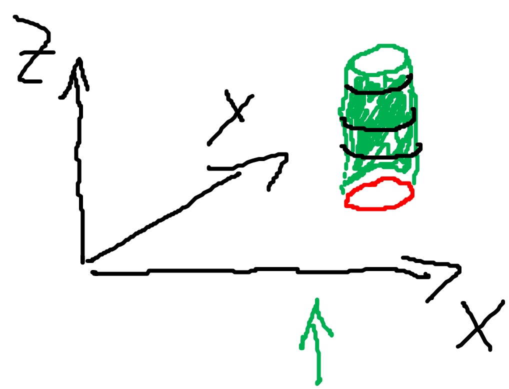
2 projections



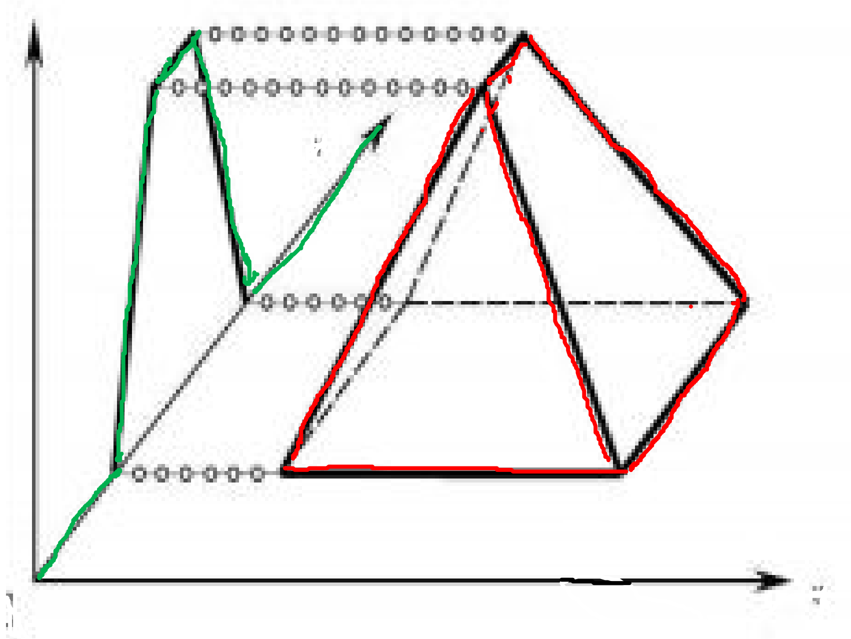
6 projections

Cylindrical Extension

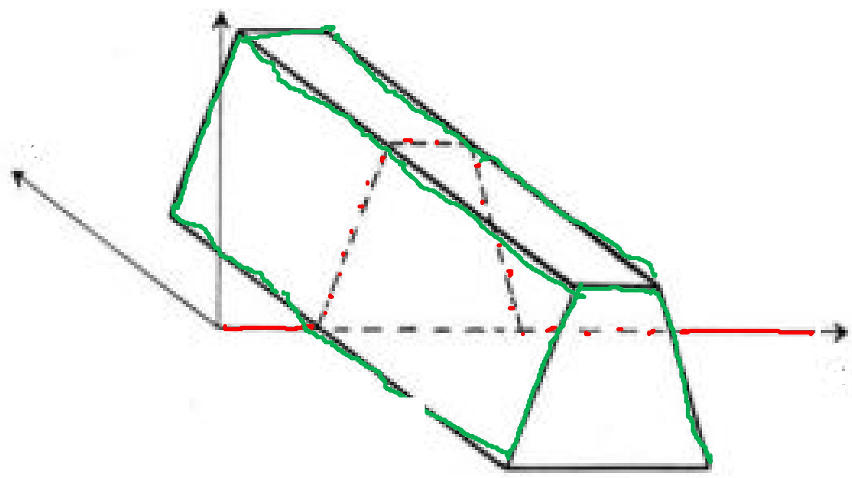




projection of μ



cylindrical extension of μ



Projection

Given a relation $R(x_1, \dots, x_n)$.

Let $[R \downarrow \mathcal{Y}]$ denote the **projection** of R on $\mathcal{Y}$.

It disregards all sets in X except those in the family

$$\mathcal{Y} = \{X_j \mid j \in J \subseteq \mathbb{N}_n\} .$$

Then $[R \downarrow \mathcal{Y}]$ is a fuzzy relation whose membership function is defined on the Cartesian product of the sets in $\mathcal{Y}$

$$[R \downarrow \mathcal{Y}](\boldsymbol{y}) = \max_{\boldsymbol{x} \succ \boldsymbol{y}} R(\boldsymbol{x}).$$

Under special circumstances, this projection can be generalized by replacing the max operator by another t-conorm.

Example

Consider the sets $X_1 = \{0, 1\}$, $X_2 = \{0, 1\}$, $X_3 = \{0, 1, 2\}$
and the ternary fuzzy relation on $X_1 \times X_2 \times X_3$:

Let $R_{ij} = [R \downarrow \{X_i, X_j\}]$ and $R_i = [R \downarrow \{X_i\}]$ for all $i, j \in \{1, 2, 3\}$.

Using this notation, all possible projections of R are given below.

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$	$R_{13}(x_1, x_3)$	$R_{23}(x_2, x_3)$	$R_1(x_1)$	$R_2(x_2)$	$R_3(x_3)$
0	0	0	0.4	0.9	1.0	0.5	1.0	0.9	1.0
0	0	1	0.9	0.9	0.9	0.9	1.0	0.9	0.9
0	0	2	0.2	0.9	0.8	0.2	1.0	0.9	1.0
0	1	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0	1	1	0.0	1.0	0.9	0.5	1.0	1.0	0.9
0	1	2	0.8	1.0	0.8	1.0	1.0	1.0	1.0
1	0	0	0.5	0.5	0.5	0.5	1.0	0.9	1.0
1	0	1	0.3	0.5	0.5	0.9	1.0	0.9	0.9
1	0	2	0.1	0.5	1.0	0.2	1.0	0.9	1.0
1	1	0	0.0	1.0	0.5	1.0	1.0	1.0	1.0
1	1	1	0.5	1.0	0.5	0.5	1.0	1.0	0.9
1	1	2	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Example

The projection R_{12}:

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$
0	0	0	0.4	
0	0	1	0.9	$(0,0) \mapsto$ max[R(0, 0, 0), R(0, 0, 1), R(0, 0, 2)] = 0.9
0	0	2	0.2	
0	1	0	1.0	
0	1	1	0.0	$(0,1) \mapsto$ max[R(0, 1, 0), R(0, 1, 1), R(0, 1, 2)] = 1.0
0	1	2	0.8	
1	0	0	0.5	
1	0	1	0.3	$(1,0) \mapsto$ max[R(1, 0, 0), R(1, 0, 1), R(1, 0, 2)] = 0.5
1	0	2	0.1	
1	1	0	0.0	
1	1	1	0.5	$(1,1) \mapsto$ max[R(1, 1, 0), R(1, 1, 1), R(1, 1, 2)] = 1.0
1	1	2	1.0	

Cylindrical Extension

Another operation on relations is called **cylindric extension**.

Let $\mathcal{X}$ and $\mathcal{Y}$ denote the same families of sets as used for projection.

Let R be a relation defined on Cartesian product of sets in family $\mathcal{Y}$.

Let $[R \uparrow \mathcal{X} \setminus \mathcal{Y}]$ denote the cylindric extension of R into sets X_1, $(i \in \mathbb{N}_n)$ which are in $\mathcal{X}$ but not in $\mathcal{Y}$.

It follows that for each $\boldsymbol{x}$ with $\boldsymbol{x} \succ \boldsymbol{y}$

$$[R \uparrow \mathcal{X} \setminus \mathcal{Y}](\boldsymbol{x}) = R(\boldsymbol{y}).$$

The cylindric extension

- produces **largest** fuzzy relation that is compatible with projection,
- is the least specific of all relations compatible with projection,
- guarantees that no information not included in projection is used to determine extended relation.

Example

Consider again the example for the projection.

The membership functions of the cylindric extensions of all projections are already shown in the table under the assumption that their arguments are extended to (x_1, x_2, x_3) *e.g.*

$$[R_{23} \uparrow \{X_1\}](0,0,2) = [R_{23} \uparrow \{X_1\}](1,0,2) = R_{23}(0,2) = 0.2.$$

In this example none of the cylindric extensions are equal to the original fuzzy relation.

This is identical with the respective projections.

Some information was lost when the given relation was replaced by any one of its projections.

Binary Fuzzy Relations

Basic Notions

Binary relations are significant among n-dimensional relations.

They are (in some sense) generalized mathematical functions.

On the contrary to functions from X to Y, binary relations $R(X, Y)$ may assign to each element of X two or more elements of Y.

Some basic operations on functions, *e.g.* inverse and composition, are applicable to binary relations as well.

Given a fuzzy relation $R(X, Y)$.

Its **domain** $\operatorname{dom} R$ is the fuzzy set on X whose membership function is defined for each $x \in X$ as

$$\operatorname{dom} R(x) = \max_{y \in Y}\{R(x, y)\},$$

i.e. each element of X belongs to the domain of R to a degree equal to the strength of its strongest relation to any $y \in Y$.

Basic Notions

The **range** ran of $R(X, Y)$ is a fuzzy relation on Y whose membership function is defined for each $y \in Y$ as

$$\text{ran } R(y) = \max_{x \in X}\{R(x, y)\},$$

i.e. the strength of the strongest relation which each $y \in Y$ has to an $x \in X$ equals to the degree of membership of y in the range of R.

The **height** h of $R(X, Y)$ is a number defined by

$$h(R) = \max_{y \in Y} \max_{x \in X}\{R(x, y)\}.$$

$h(R)$ is the largest membership grade obtained by any pair $(x, y) \in R$.

Representation and Inverse

Consider *e.g.* the **membership matrix** $\boldsymbol{R} = [r_{xy}]$ with $r_{xy} = R(x,y)$.

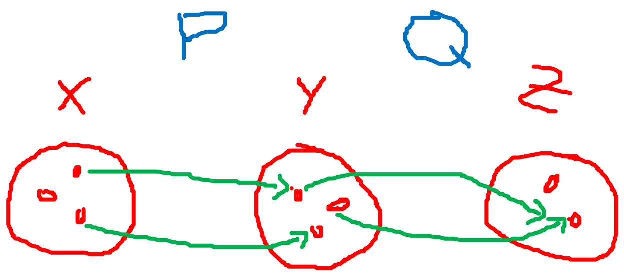
Its **inverse** $R^{-1}(Y,X)$ of $R(X,Y)$ is a relation on $Y \times X$ defined by

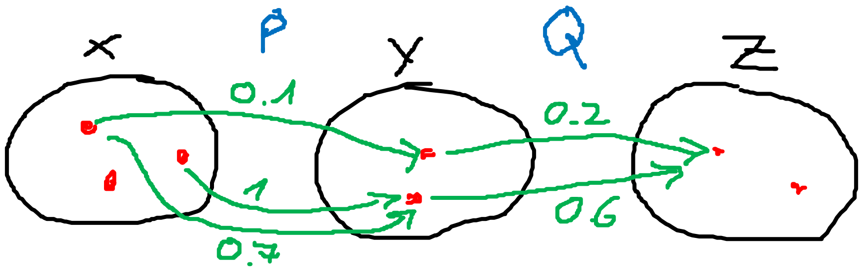
$$R^{-1}(y,x) = R(x,y) \quad \text{for all } x \in X,\, y \in Y.$$

$\boldsymbol{R}^{-1} = [r_{xy}^{-1}]$ representing $R^{-1}(y,x)$ is the transpose of $\boldsymbol{R}$ for $R(X,Y)$

$$(\boldsymbol{R}^{-1})^{-1} = \boldsymbol{R}$$

Standard Composition





Consider the binary relations $P(X,Y)$, $Q(Y,Z)$ with common set Y.

The **standard composition** of P and Q is defined as

$$(x,z) \in P \circ Q \iff \exists y \in Y : \{(x,y) \in P \wedge (y,z) \in Q\}.$$

In the fuzzy case this is generalized by

$$[P \circ Q](x,z) = \sup_{y \in Y} \min\{P(x,y),\ Q(y,z)\}.$$

If Y is finite, sup operator can be replaced by max.

The standard composition is also called **max-min composition**.

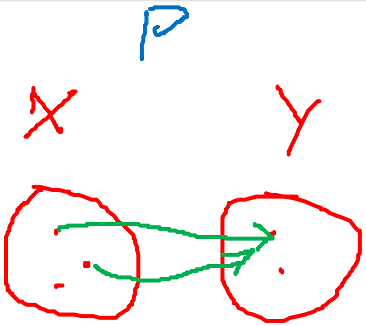
Example

$$P \circ Q = R$$

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & 5 & .7 \\ .5 & .4 & .5 & .5 \end{bmatrix}$$

$$\begin{aligned} r_{11} &= \max\{\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})\} \\ &= \max\{\min(.3, .9), \min(.5, .3), \min(.8, 1)\} \\ &= .8 \\ r_{32} &= \max\{\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})\} \\ &= \max\{\min(.4, .5), \min(.6, .2), \min(.5, 0)\} \\ &= .4 \end{aligned}$$

Inverse of Standard Composition



The inverse of the max-min composition follows from its definition:

$$[P(X,Y)\circ Q(Y,Z)]^{-1} = Q^{-1}(Z,Y)\circ P^{-1}(Y,X).$$

Its associativity also comes directly from its definition:

$$[P(X,Y)]\circ Q(Y,Z)]\circ R(Z,W) = P(X,Y)\circ [Q(Y,Z)\circ R(Z,W)].$$

Note that the standard composition is not commutative.

Matrix notation: $[r_{ij}] = [p_{ik}]\circ [q_{kj}]$ with $r_{ij} = \max_k \min(p_{ik}, q_{kj})$.

Example: Properties of Airplanes (Speed, Height, Type)

4 possible speeds: s_1, s_2, s_3, s_4
3 heights: h_1, h_2, h_3
2 types: t_1, t_2

Consider the following fuzzy relations for airplanes:

- relation *A* between speed and height,
- relation *B* between height and the type.

A	h_1	h_2	h_3
s_1	1	.2	0
s_2	.1	1	0
s_3	0	1	1
s_4	0	.3	1

B	t_1	t_2
h_1	1	0
h_2	.9	1
h_3	0	.9

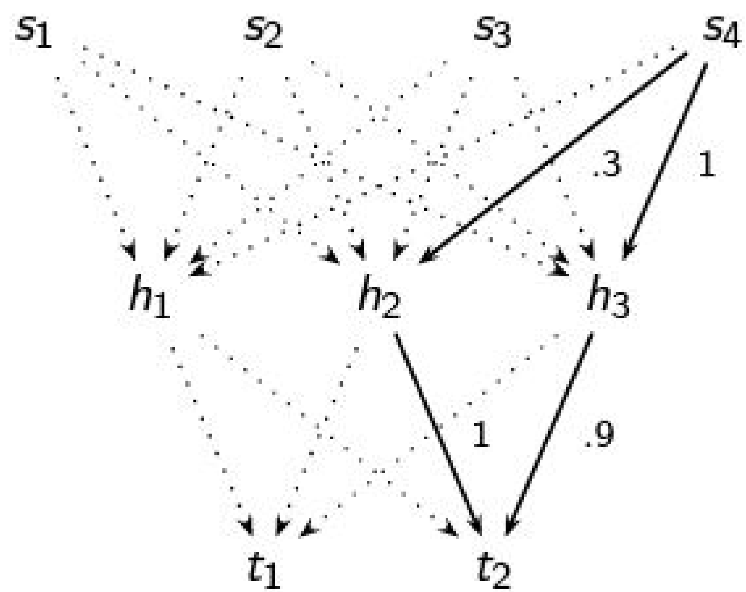
Example: Properties of Airplanes (Speed, Height, Type)

matrix multiplication scheme

$$
\begin{array}{ccc|cc}
 & & & 1 & 0 \\
\boldsymbol{A} & \circ & \boldsymbol{B} & .9 & 1 \\
 & & & 0 & .9 \\
\hline
1 & .2 & 0 & 1 & .2 \\
.1 & 1 & 0 & .9 & 1 \\
0 & 1 & 1 & .9 & 1 \\
0 & .3 & 1 & .3 & .9
\end{array}
$$

$A \circ B$ speed-type relation

flow scheme



$$
\begin{aligned}
(A \circ B)(s_4, t_2) &= \max\{\min\{.3,\ 1\},\ \min\{1,\ .9\}\} \\
&= .9
\end{aligned}
$$

Binary Relations on a Single Set

It is also possible to define crisp or fuzzy binary relations among elements of a single set X.

Such a binary relation can be denoted by $R(X,X)$ or $R(X^2)$ which is a subset of $X \times X = X^2$.

These relations are often referred to as **directed graphs** which is also ar representation of them.

- Each element of X is represented as node.
- Directed connections between nodes indicate pairs of $x \in X$ for which the grade of the membership is nonzero.
- Each connection is labeled by its actual membership grade of the corresponding pair in R.

Example

An example of $R(X, X)$ defined on $X = \{1, 2, 3, 4\}$.

Two different representation are shown below.

	1	2	3	4
1	.7	0	.3	0
2	0	.7	1	0
3	.9	0	0	1
4	0	0	.8	.5